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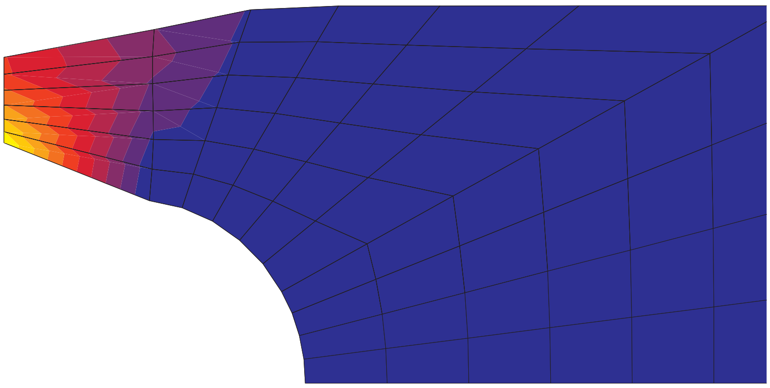

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Numerical Implementation Of An Isotropic Finite Strain Elasto-Plastic Material Model Formulated in Principal Axes

Master Thesis – February 2005

Abstract

This work is concerned with the theory and numerical implementation of a finite strain plasticity model considering isotropic hardening. We will consider a certain type of a finite strain plasticity model for isotropic elasto-plasticity. This model is formulated in terms of the elastic left CAUCHY-GREEN-tensor $\mathbf{b}^e$. The evolution of $\mathbf{b}^e$ is accomplished by means of an exponential map. The proposed material model allows isotropic hyper-elastic laws as well as isotropic yield conditions. Thus the task was to add certain subroutines to an existing material model referred to as tensor model. The model is first formulated in a tensor related representation and is then one-to-one translated into principal axes. The algorithmic elasto-plastic tangent operator is obtained in closed form, a task which has been simplified by application of a particular eigenvalue formalism in contrast to tensor formalism. Finally, the applicability of the proposed eigenvalue model and of the tensor model is demonstrated by three representative examples, in which both model results are compared.

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Bochum, February 2005 **Pembarthi Madhu**

Contents

Variable definitions

θ^α, θ^3	Convective coordinates
$\mathbf{C}$	Right Cauchy-Green tensor
$\mathbf{E}$	Green-Lagrange strain tensor
$\mathbf{i}_i = \mathbf{i}^i$	Orthogonal unit vectors
$\mathbf{d}$	Director ($\|\|\mathbf{d}\|\| = 1$)
$\mathbf{G}, \mathbf{g}$	Metric tensors
$\mathbf{G}_i, \mathbf{g}_i$	Base vectors
$\mathbf{b}^{-1}$	Inverse left Cauchy-Green tensor
$\mathbf{e}$	Almansi strain tensor

Tensor operations, symbols:

$\mathbf{A^T}$, $\mathbf{A}^*$, $\mathbf{A^{-1}}$	Transpose, dual and inverse tensor
$\mathbf{a} \cdot \mathbf{b}$	Simple contraction of two vectors
$\mathbf{A} \cdot \mathbf{B}$	Simple contraction of two tensors
$\mathbf{A} : \mathbf{B}$	Double contraction of two tensors
$\mathbf{a} \times \mathbf{b}$	Dyadic product of two vectors
$\mathbf{A} \otimes \mathbf{B}$	Dyadic Product of two tensors
$\mathbf{I}$	Identity 2-order tensor

Operation

tr	trace ($\mathrm{tr}\mathbf{A} = \mathbf{A} : \mathbf{I}$)
dev	deviatoric part
det	Determinant
$\mathrm{Grad}, \mathrm{grad}$	material and spatial gradient
$\mathrm{Div}, \mathrm{div}$	material and spatial total divergence
D	directional derivative
L, Δ	Laplacian operator
δ	Variation
Δ	Increment

Continuum mechanics

P_0, P	Material point in the referential or current configuration
$\mathbf{X}$	Position vector of the referential configuration
X^i	Components of X in the referential configuration
$\mathbf{x}$	Position vector of the current configuration
x^i	Components of x in the current configuration

$\mathbf{G^i}$, $\mathbf{G_i}$	Contra- resp. covariant basis vectors in the referential configuration
$\mathbf{g^i}$, $\mathbf{g_i}$	Contra- resp. covariant basis vectors in the current configuration
G^{ij} , G_{ij}	Contra- resp. covariant components of the metric tensors of the referential configuration
g^{ij} , g_{ij}	Contra- resp. covariant components of the metric tensors of the current configuration
$\mathbf{G}, \mathbf{g}$	Material and spatial metric tensor
$\|G\|$, $\|g\|$	Determinants of the metric tensors
$\mathrm{d}V, \mathrm{d}v$	Material and spatial volume element
δ^i_j	KRONECKER-Delta
$\mathbf{C}, \mathbf{b}$	Right and left CAUCHY-GREEN-Tensor
$\mathbf{E}, \mathbf{e}$	GREEN-LAGRANGE- and EULER-ALMANSI-strain tensor
$\boldsymbol{\sigma}$	CAUCHY stress tensor
$\mathbf{S}$	2. PIOLA-KIRCHHOFF stress tensor
$\mathbf{P}$	1. PIOLA-KIRCHHOFF stress tensor
$\boldsymbol{\tau}$	KIRCHHOFF stress tensor
$\delta W, \delta w$	Virtual work in material and spatial discription
$(\cdot)^{int}, (\cdot)^{ext}$	Inner and external part of the energy and work expression
$\mathbf{D}, \mathbf{d}$	Undeformed and deformed shell director of 1 order
λ	Thickness stretch

Material Model

W	Energy function
ψ	Free HELMHOLTZ-Energy
E	Elasticity modulus
ν	Poisson ratio
κ	Bulk modulus
μ	Shear modulus

Chapter 1

Preliminaries

1.1 Introduction

The theory of elastoplastic media is now a mature branch of solid and structural mechanics, having experienced significant development during the latter half of this century. In particular, the classical theory, which deals with small-strain elastoplasticity problems, has a firm mathematical basis and from this basis further developments, both mathematical and computational, have evolved. Small-strain elastoplasticity is well understood and the understanding of its governing equations can be said to be almost complete. Likewise, theoretical, computational and algorithmic work on approximations of desired accuracy can be achieved with confidence.

The finite-strain theory has evolved along parallel lines, although it is considerably more complex and is subject to a number of alternative treatments. The form taken by the governing equations is reasonably settled, though there is as yet no mathematical treatment of existance, uniqueness and stability analogous to those of the small-strain case. Computationally, great strides have been made in the last two decades and it is now possible to solve highly complex problems with the aid of the computer.

1.2 Previous studies

It is generally agreed that the origin of plasticity dates back to a series of papers by Tresca from 1864 to 1872 on the extrusion of metals. In this work the first yield condition was proposed : The condition, known subsequently as the Tresca yield criterion, stated that a metal yields when the maximal shear stress attains a critical value. In the same time period, St.Venant introduced basic constitutive relations for rigid, perfectly plastic materials in plane stress and suggested that principal axes of the strain increment coincide with the principal axes of stress. Lévy derived the general equations in three dimensions. In 1913, [von Mises 1913] derived the general equations for plasticity, accompanied by his well-known pressure-insensitive yield criterion(J_2-theory, or octahedral shear stress yield condition).

In 1924, [Prandtl 1924] extended the St.Venant-Levy-von Mises equations for the plane continuum problem to include the elastic component of strain and [Reuss 1939] in 1930 carried out their extension to three dimensions.

Compared to perfect plasticity, the developement of incremental constitutive relations for hardening materials proceeded more slowly. In 1928, [Prandtl 1928] attempted to formulate general relations for hardening behavior. In 1938, [Melan 1938] generalized the foregoing concepts of perfect plasticity by giving incremental relations for hardening solids with smooth yield surface and discussing uniqueness results for elastoplastic incremental problems for both perfectly plastic and hardening materials, based on some limiting assumptions.

Since 1940, the theory of plasticity has seen relatively more rapid developement. A detailed description of early development of plasticity theory and a comprehensive list of references on plasticity published before 1980 can be found in [Życzkowski 1981].

Trying to close the gap from the early beginnings to the present situation is almost impossible. Since then the development of plasticity models, especially finite strain plasticity models, was considerably rapid. Today this subject in particular in the isotropic case is very much understood. Two important monographs which study this problems in a variety of ways are [Simo & Hughes 1998] and [Han & Reddy 1999]. The book of [Han & Reddy 1999] focusses in particular on a rigourous mathematical analysis of plasticity whereas in [Simo & Hughes 1998] a comprehensive account of the field of linear and nonlinear elasto-plasticity can be found in a modern style of nonlinear continuum mechanics. For more detail of the current state of plasticity these two monographs are recommended.

1.3 Objective of this work

The problem when dealing with finite strain plasticity modelling is that each type of model has to be considered on a case by case basis in contrast to infinitesimal plasticity since the nonlinear case opens a wide range of new questions like the appropriate choice of strain and stress tensors or a suitable choice of objective time derivatives.

In this work we will consider a certain type of a finite strain plasticity model for isotropic elasto-plasticity. This model is formulated in terms of the elastic left CAUCHY-GREEN-tensor $\mathbf{b}^e$. The evolution of $\mathbf{b}^e$ is accomplished by means of an exponential map. Apart from this the model is similar to one proposed in [SIMO & HUGHES 1998]. However, in the present form, due to the exponential map and since we donnot use logarithmic strains, we cannot exploit the fact that the normal to the yield surface is defined by the trial normal at the beginning of the Return map. Therefore to increase accuracy the normal has to be included in the nonlinear equation system as independent residual. This, however, makes the algorithm a bit more complex. Thus, to increase efficiency we strive to formulate this model in principal axes, where we can exploit the fact that the eigenprojections of $\mathbf{b}^e$ are invariant during the Return Map. This results in a 4-dimensional problem, where besides the consistency parameter the three eigenvalues of the normal to the yield surface are independent variables.

To conclude, the model relies on the following assumptions: (i) the isotropy of the hyperelastic law, (ii) the use of $\mathbf{b}^e$ as driving variable, (iii) the coaxiality of the KIRCHHOFF-stress measure $\boldsymbol{\tau}$ and $\mathbf{b}^e$, (iv) the use of the usual VON MISES-J_2-yield function, (v) the evolution of $\mathbf{b}^e$ by means of an exponential map, (vi) the consideration of the normal to the yield surface as independent variable.

The model is first formulated in a tensor related representation and is then one-to-one translated into principal axes. Thereby the dimensionality of the problem is reduced from 10 to 4.

1.4 Analysing Tools

MARC-MENTAT program has been recognized as the premier general purpose program for nonlinear finite element analysis since the mid-1970s. This system contains a series of integrated programs that facilitate analysis of engineering problems in the fields of structural mechanics, heat transfer and electro magnetics. Since the mid-1970s, MARC has been recognized as the premier general purpose program for nonlinear finite element analysis

The MARC system consists of the following programs:

- MARC
- MENTAT

These programs work together to:

- Generate geometric information that defines your structure (MARC and MSC.MARC MENTAT (MENTAT))
- Analyze your structure (MARC)
- Graphically depict the results (MARC and MENTAT)

1.4.1 MARC

MARC can be used to perform linear or nonlinear stress analysis in the static and dynamic regimes and to perform heat transfer analysis. Physical problems in one, two, or three dimensions can be modeled using a variety of elements. These elements include trusses, beams, shells and solids. Mesh generators, graphics and post processing capabilities, which assist you in the preparation of input and the interpretation of results, are all available in MARC.

1.4.2 MENTAT

MENTAT is an interactive computer program that prepares and processes data for use with the finite element method. Interactive computing can significantly reduce the human effort needed for analysis by the finite element method. Graphical presentation of data further reduces this effort by providing an effective way to review the large quantity of data typically associated with finite element analysis.

An important aspect of MENTAT is that you can interact directly with the program. MENTAT verifies keyboard input and returns recommendations or warnings when it detects questionable input. MENTAT checks the contents of input files and generates warnings about its interpretation of the data if the program suspects that it may not be processing the data in the manner in which you, the user, have assumed. MENTAT allows you to graphically verify any changes.

MENTAT can process both two- and three-dimensional meshes to do the following: Generate and display a mesh and display boundary conditions and loadings perform post-processing to generate contour, deformed shape and time history plots.

Chapter 2

Kinematics

2.1 General background

Any nonrigid body deforms when it is subjected to external forces. The deformation is called *elastic* if it is reversible and time independent, that is, if the deformation vanishes instantaneously as soon as forces are removed. A reversible but time-dependent deformation is known as *viscoelastic*; in this case the deformation increases with time after application of load and it decreases slowly after the load is removed. The deformation is called *plastic* if it is irreversible or permanent.

In modeling the material nonlinear behavior of solids, plasticity theory is applicable primarily to those bodies that can experience inelastic deformations considerably greater than the elastic deformation. If the resulting total deformation, including both translations and rotations, are small enough, we can apply small deformation theory in solving these problems. If, however strains and rotations are finite, one must resort to the theory of large deformations. In doing so, we will be using two sets of representations, namely: *Material coordinates* in the undeformed or reference configuration, also called *Lagrangian coordinates* and *Spatial coordinates* in the deformed or current configuration, also called *Eulerian coordinates*.

2.2 Deformation

In finite strain elasticity we distinguish two configurations, the reference and current configuration. The deformation assigns to each point in the reference configuration $\mathbf{X}$ its deformed image $\mathbf{x}$ in the current configuration.

2.2.1 Lagrangian and Eulerian coordinates

Lagrangian coordinates. We consider the configuration B_0 of a body at time t_0 in a 3D Euclidean space E3. In B_0, the body is supposed to be unloaded, undeformed and unstressed. This is set as the initial configuration at the beginning of the computation. The position vector of a typical point P_0 of B_0 relative to the origin O of an orthogonal Cartesian coordinate system is denoted by

$$\mathbf{X} = X^i \mathbf{i}_i \,, \tag{2.1}$$

where X^i are LAGRANGIAN or *material coordiantes* and $\mathbf{i}_i = \mathbf{i}^i$ are unit vectors along X_i-axes **(Figure.2.1)**.

Eulerian coordinates: We now suppose the body to take at a certain time t a new configuration B in E3, due to the action of external forces. This is set as the current configuration. Then, the point P_0 is moved into the position P which will be determined with respect to the same origin O by the position vector

$$\mathbf{x} = x^i \mathbf{i}_i \,, \tag{2.2}$$

where x^i are called EULERIAN or *spatial coordinates*. The two position vectors are connected by the displacement $\mathbf{u}$. In vector algebra the deformation relationship reads as:

$$\mathbf{x} \;=\; \mathbf{X} + \mathbf{u} \,, \tag{2.3}$$

where $\mathbf{X}$ is the position vector in the reference configuration and $\mathbf{x}$ is the position vector in the current configuration. Here, we resolve each vector with respect to a global reference frame, that is:

$$x^i \mathbf{i}_i = X^k \mathbf{i}_k + u^m \mathbf{i}_m \,, \tag{2.4}$$

with the components x^i, X^k, u^m and an orthonormal base vector system $\mathbf{i}_i$. If we consider the **Equation (2.3)** in differential form we obtain :

$$d\mathbf{x} = \mathbf{F}\, d\mathbf{X} \,. \tag{2.5}$$

2.3 Deformation gradient

In contrast to the additive relation of **Equation** (2.3) we now have a multiplicative relationship. $\mathbf{F}$ denotes a second-order tensor which maps the differential element $d\mathbf{X}$ onto $d\mathbf{x}$. The quantity $\mathbf{F}$ is crucial in nonlinear continuum mechanics and is a primary measure

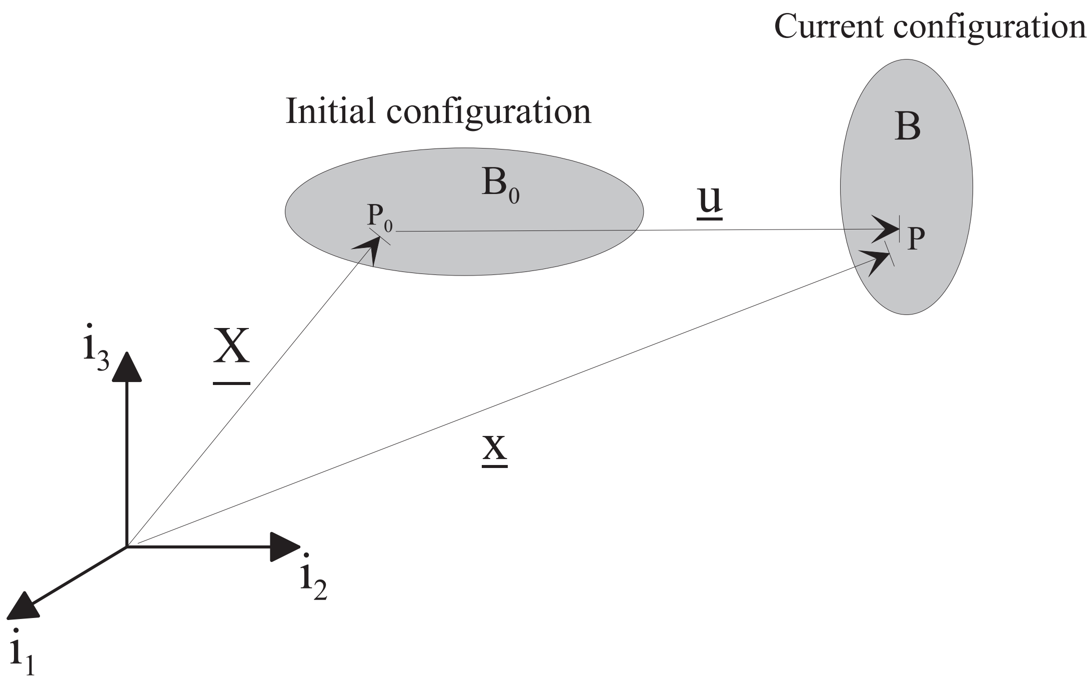

Figure **2.1**: Material and spatial coordinates; displacement vector **u**

of deformation called the deformation gradient characterising the motion behaviour in the infinitesimal neighbourhood of a point. The definition for **F** is given as :

$$\mathbf{F} = \mathbf{I} + \mathrm{Grad}\,\mathbf{u} = \mathbf{I} + \mathbf{H}\,. \tag{2.6}$$

Here the gradient of the displacement vector $\mathbf{H} = \mathrm{Grad}\,\mathbf{u}$ has been used. Considering the linear strain measure ϵ the gradient $\mathbf{H}$ is employed in the form

$$\epsilon = \frac{1}{2}(\mathbf{H} + \mathbf{H}^T) = \frac{1}{2}(\mathrm{Grad}\,\mathbf{u} + (\mathrm{Grad}\,\mathbf{u})^T)\,. \tag{2.7}$$

Since the gradient is defined by

$$\mathrm{Grad}\,\mathbf{u} = \frac{\partial \mathbf{u}}{\partial \mathbf{X}} = \frac{\partial \mathbf{u}}{\partial X^i}\mathbf{i}_i = \frac{\partial u^j}{\partial X^i}\mathbf{i}_j \otimes \mathbf{i}_i \;\Rightarrow\; (\mathrm{Grad}\,\mathbf{u})^T = \frac{\partial u^j}{\partial X^i}\mathbf{i}_i \otimes \mathbf{i}_j\,, \tag{2.8}$$

we get in particular the following components of ϵ:

$$\epsilon_{11} = \frac{\partial u^1}{\partial X^1}\,, \tag{2.9}$$

$$\epsilon_{12} = \frac{1}{2}(\frac{\partial u^1}{\partial X^2} + \frac{\partial u^2}{\partial X^1})\,, \cdots \tag{2.10}$$

2.4 Strain tensors

We will now introduce second-order strain tensors which measure length and angle changes of differential elements during a motion. There are numerous definitions of strain tensors proposed in literature, however, here we will introduce only the most important ones, the so called Green-Lagrange and Almansi-strain tensors. They are called objective, since they measure only relative length and angle changes and vanish for pure rigid body movements.

2.4.1 The right and left Cauchy-Green tensors

At first, we will introduce the right Cauchy-Green tensor $\mathbf{C}$ defined by

$$\mathbf{C} = \mathbf{F}^* \, \mathbf{g} \, \mathbf{F} \,, \tag{2.11}$$

where $\mathbf{g}$ is the spatial metric. $\mathbf{C}$ is symmetric and positive-definite and, therefore

$$\mathbf{C} = \mathbf{F}^* \, \mathbf{g} \, \mathbf{F} = (\mathbf{F}^* \, \mathbf{g} \, \mathbf{F})^* = \mathbf{C}^* \,. \tag{2.12}$$

It is obvious that all six components of $\mathbf{C}$ can be computed, if all nine components of $\mathbf{F}$ are given, but it is impossible to compute $\mathbf{F}$ if only $\mathbf{C}$ is given. A further important strain measure is the left Cauchy-Green tensor $\mathbf{b}$ defined by

$$\mathbf{b} = \mathbf{F} \, \mathbf{G}^{-1} \, \mathbf{F}^* \,, \tag{2.13}$$

where $\mathbf{G}^{-1}$ is the inverse metric of the initial state. The second-order tensor $\mathbf{b}$ is like $\mathbf{C}$ symmetric and positive-definite

$$\mathbf{b} = \mathbf{F} \, \mathbf{G}^{-1} \, \mathbf{F}^* = (\mathbf{F} \, \mathbf{G}^{-1} \, \mathbf{F}^*)^* = \mathbf{b}^* \,. \tag{2.14}$$

2.4.2 Polar decomposition

The deformation gradient $\mathbf{F}$ can be uniquely decomposed into a rotational and a stretching part, as shown in **(Figure 2.2)**. Depending on the sequence , if we first stretch the body and then rotate it to the final position we have the decomposition

$$\mathbf{F} = \mathbf{R}\mathbf{U} \tag{2.15}$$

and, for the second case, if we rotate first and then stretch, we obtain

$$\mathbf{F} = \mathbf{v}\mathbf{R}\,. \tag{2.16}$$

In the above equation, the quantity $\mathbf{R}$ is a proper orthogonal tensor with $\mathbf{R}^T = \mathbf{R}^{-1}$ and

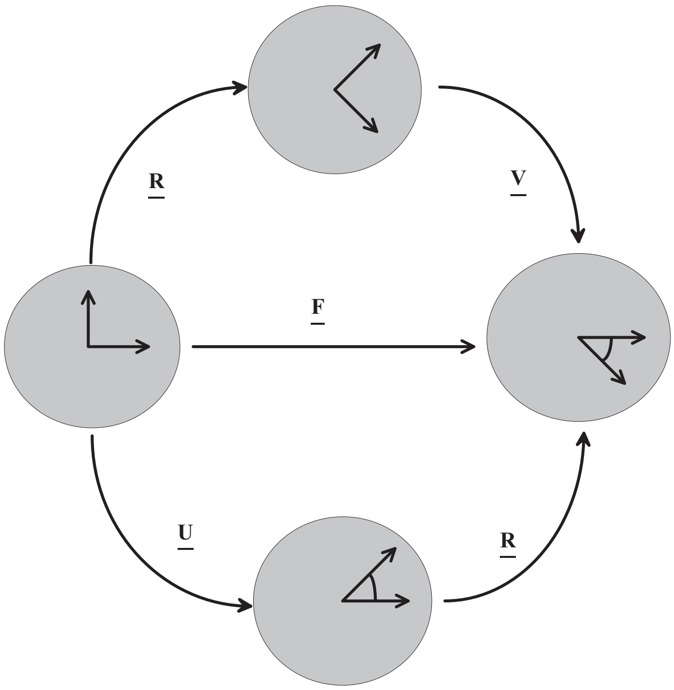

Figure **2.2**: Decomposition in rotational and stretching part

is called a *rotation tensor*. $\mathbf{U}$ and $\mathbf{v}$ define unique, positive-definite, symmetric tensors which are called the **right (or material) stretch tensor** and the **left (or spatial) stretch tensor**, respectively. It can be shown that the right Cauchy-Green tensor $\mathbf{C}$ can be expressed as

$$\mathbf{C} = \mathbf{F}^{*}\,\mathbf{g}\,\mathbf{F} = (\mathbf{R}\mathbf{U})^{*}\,\mathbf{g}\,\mathbf{R}\mathbf{U} = \mathbf{U}^{*}\mathbf{R}^{*}\,\mathbf{g}\,\mathbf{R}\mathbf{U} = \mathbf{G}\,\mathbf{U}^{2}\,, \tag{2.17}$$

because $\mathbf{U}$ is symmetric ($\mathbf{U}^{*} = \mathbf{G}\,\mathbf{U}\,\mathbf{G}^{-1}$) and $\mathbf{R}$ is proper orthogonal and therefore $\mathbf{R}^{*}\,\mathbf{g}\,\mathbf{R} = \mathbf{G}$. Likewise, for the spatial left Cauchy-Green tensor $\mathbf{b}$ we get

$$\mathbf{b} = \mathbf{F}\,\mathbf{G}^{-1}\,\mathbf{F}^{*} = \mathbf{v}\mathbf{R}\,\mathbf{G}^{-1}\,(\mathbf{v}\mathbf{R})^{*} = \mathbf{v}\mathbf{R}\,\mathbf{G}^{-1}\,\mathbf{R}^{*}\mathbf{v}^{*} = \mathbf{v}^{2}\,\mathbf{g}^{-1}\,, \tag{2.18}$$

with $\mathbf{v}^{*} = \mathbf{g}\,\mathbf{v}\,\mathbf{g}^{-1}$.

2.4.3 Green-Lagrange strain tensor

These two tensors are suitable to construct nonlinear strain measures. A useful strain measure called Green-Lagrange strain tensor is defined by:

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{G}) \,, \tag{2.19}$$

where $\mathbf{G}$ is a certain representation of $\mathbf{I}$, the identity tensor of second-order. We consider $\mathbf{E}$ to be nonlinear, since

$$\begin{aligned} \mathbf{E} &= \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I}) = \frac{1}{2}((\mathbf{I} + \mathbf{H})^T(\mathbf{I} + \mathbf{H}) - \mathbf{I}) \,, \\ &= \frac{1}{2}(\mathbf{H} + \mathbf{H}^T + \mathbf{H}^T\mathbf{H} + \mathbf{I} - \mathbf{I}) \,, \\ &= \boldsymbol{\epsilon} + \frac{1}{2}(\mathbf{H}^T\mathbf{H}) \quad . \end{aligned} \tag{2.20}$$

2.5 Metric Properties

In this section we introduce *co-variant* and *contra-variant* base vectors and discuss the construction of metric tensor components.

2.5.1 Co-variant base vectors

Consider a special coordinate θ^i, which is not only curvilinear but, in addition, convective. The corresponding definitions for the co-variant base vectors are given as follows:

$$\mathbf{G}_i = \frac{\partial \mathbf{X}}{\partial \theta^i} \,, \qquad \mathbf{g}_i = \frac{\partial \mathbf{x}}{\partial \theta^i} \,. \tag{2.21}$$

Convective means that the coordinate system is inscribed on the body. At each point P of a body the base vectors $\mathbf{G}_i$ are by virtue of **Equation** (2.21) tangential to the coordinate line θ^i at the point $\mathbf{X}$ located in the initial configuration. And analogously for $\mathbf{g}_i$, which are tangential to the coordinate lines θ^i at the point $\mathbf{x}$. A direct consequence is that at each material point the coordinates θ^i are the same in any configuration considered. The coordinate lines are somehow fixed to the material point and deform with the material in contrast to a basis which is fixed in space like $\mathbf{i}_i$. Using **Equations** (2.21) and (2.5) we are able to obtain a relationship between both base vector systems:

$$\mathbf{g}_i = \frac{\partial \mathbf{x}}{\partial \theta^i} = \frac{d\mathbf{x}}{d\theta^i} = \frac{\mathbf{F}d\mathbf{X}}{d\theta^i} = \mathbf{F}\frac{\partial \mathbf{X}}{\partial \theta^i} = \mathbf{F}\mathbf{G}_i \,, \tag{2.22}$$

that means $\mathbf{g}_i$ and $\mathbf{G}_i$ are coupled by means of the deformation gradient $\mathbf{F}$.

2.5.2 Contra-variant base vectors

We can obtain the contra-variant base vectors $\mathbf{G}^i$ and $\mathbf{g}^i$ as follows:

- Considering the position vectors $\mathbf{X}$ and $\mathbf{x}$, the co-variant base vectors are obtained using **Equation** (2.21).
- Compute the metric tensor components

$$G_{ij} = \mathbf{G}_i \cdot \mathbf{G}_j = \mathbf{G}_j \cdot \mathbf{G}_i = G_{ji}\,, \tag{2.23}$$

$$g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j = \mathbf{g}_j \cdot \mathbf{g}_i = g_{ji}\,. \tag{2.24}$$

- Invert the co-variant metric tensor components

$$[G^{ij}] = [G_{ij}]^{-1}\,, \qquad [g^{ij}] = [g_{ij}]^{-1}\,. \tag{2.25}$$

- Raising the indices of $\mathbf{G}_i$ and $\mathbf{g}_i$ by means of the metric

$$\mathbf{G}^i = G^{ij}\mathbf{G}_j\,, \qquad \mathbf{g}^i = g^{ij}\mathbf{g}_j\,. \tag{2.26}$$

As a well known fact for orthonormal coordinates the relation $\mathbf{i}_i \cdot \mathbf{i}_j = \delta_{ij}$ holds which has its analogy in:

$$\mathbf{G}_i \cdot \mathbf{G}^j = \delta_i^j\,, \tag{2.27}$$

$$G_{ij}G^{jk} = G^{kj}G_{ji} = \delta_i^k\,, \qquad g_{ij}g^{jk} = g^{kj}g_{ji} = \delta_i^k\,, \tag{2.28}$$

which were already used in **Equation** (2.26). δ_i^j is called KRONECKER-delta and represents the following matrix:

$$\delta_j^i = \delta_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.29}$$

By introducing convective coordinates we obtain a very simple expression for the components of $\mathbf{E}$:

$$\mathbf{E} = E_{ij}\mathbf{G}^i \otimes \mathbf{G}^j = \frac{1}{2}(g_{ij} - G_{ij})\mathbf{G}^i \otimes \mathbf{G}^j\,, \tag{2.30}$$

that means to compute the strains we simply have to subtract the metric tensor components related to the initial and current configuration from each other. The metric measures in fact lengths and angles. For example if we consider the product of two vectors $\mathbf{U} \cdot \mathbf{V}$ we get:

$$\mathbf{U} \cdot \mathbf{V} = U^i \mathbf{G}_i \cdot V^j \mathbf{G}_j = U^i V^j \mathbf{G}_i \cdot \mathbf{G}_j = U^i V^j G_{ij} \,. \tag{2.31}$$

Similar holds for the product of two vectors which are related to the current configuration. Using **Equation** (2.5) $\mathbf{E}$ can be also defined by:

$$d\mathbf{X} \cdot \mathbf{E}\, d\mathbf{X} = \frac{1}{2}(d\mathbf{x} \cdot \, \mathbf{g}\, d\mathbf{x} \; - d\mathbf{X} \cdot \, \mathbf{G}\, d\mathbf{X}) \,, \tag{2.32}$$

that means the Green-Lagrange strain measures the length and angle changes of two differential line elements belonging to the same material point P but coupled by the deformation gradient $\mathbf{F}$. $\mathbf{E}$ can be expressed in the form **Equation** (2.19) if we use the following tensors :

$$\mathbf{C} = \mathbf{G}\,\mathbf{U}^2 = \mathbf{F}^*\, \mathbf{g}\, \mathbf{F} = C_{ij} \mathbf{G}^i \otimes \mathbf{G}^j = g_{ij} \mathbf{G}^i \otimes \mathbf{G}^j, \qquad \mathbf{G} = G_{ij} \mathbf{G}^i \otimes \mathbf{G}^j \,. \tag{2.33}$$

Now it becomes clear why we have used $\mathbf{G}$ in **Equation** (2.19). Using $\mathbf{G}_i = G_{ij} \mathbf{G}^j$ we can see that $\mathbf{G} = \mathbf{I}$ is the second-order identity tensor.

2.5.3 Almansi strain tensor

If we take the components of $\mathbf{E}$ and couple them with current base vectors we obtain a further strain measure, the so called Almansi strain tensor, which is denoted by:

$$\mathbf{e} = E_{ij}\, \mathbf{g}^i \otimes \mathbf{g}^j = \frac{1}{2}\,(g_{ij} - G_{ij})\, \mathbf{g}^i \otimes \mathbf{g}^j \,. \tag{2.34}$$

If we use the tensors

$$\mathbf{g} = g_{ij}\, \mathbf{g}^i \otimes \mathbf{g}^j, \qquad \mathbf{b}^{-1} = \mathbf{g}\, \mathbf{v}^{-2} = \mathbf{F}^{-*}\, \mathbf{G}\, \mathbf{F}^{-1} = G_{ij} \mathbf{g}^i \otimes \mathbf{g}^j \,, \tag{2.35}$$

we arrive at the following representation in absolute form:

$$\mathbf{e} = \frac{1}{2}\,(\mathbf{g} - \mathbf{b}^{-1}). \tag{2.36}$$

We also use $\mathbf{g}$ here, which is nothing else than the second-order identity tensor $\mathbf{i}$ only to emphasize that $\mathbf{e}$ has a co-variant component decomposition. The tensors $\mathbf{E}$ and $\mathbf{e}$ are in fact coupled by means of $\mathbf{F}$:

$$\mathbf{e} = \mathbf{F}^{-*}\, \mathbf{E}\, \mathbf{F}^{-1} \,. \tag{2.37}$$

If we had wished to express $\mathbf{E}$ with respect to a new basis we would have used the usual component transformation rules:

$$\bar{e}_{kl} = E_{ij}\, (\mathbf{G}^i \cdot \mathbf{g}_k)(\mathbf{G}^j \cdot \mathbf{g}_l) \neq E_{kl}. \tag{2.38}$$

2.6 Pull-back and push-forward operations

Pull-back and push-forward. *Pull-back* and *push-forward* are operations which transport the components of a tensor from the deformed basis into the undeformed one or vice versa.

2.6.1 Push-forward

The *push-forward* is symbolized by the following notation:

$$\mathbf{e} = \mathbf{F}_{\triangleright}(\mathbf{E})\,, \qquad \mathbf{g}_i = \mathbf{F}_{\triangleright}(\mathbf{G}_i)\,, \qquad \mathbf{g}^i = \mathbf{F}_{\triangleright}(\mathbf{G}^i)\,. \tag{2.39}$$

One can derive a relation similar to **Equation** (2.22) for the contra-variant base vectors $\mathbf{G}^i$ and $\mathbf{g}^i$ which reads as:

$$\begin{aligned} \mathbf{g}_i \cdot \mathbf{g}^j = \mathbf{F}\mathbf{G}_i \cdot \mathbf{g}^j = \mathbf{G}_i \cdot \mathbf{F}^*\mathbf{g}^j = \delta_i^j \quad &= \quad \mathbf{G}_i \cdot \mathbf{G}^j \Rightarrow \mathbf{G}^j = \mathbf{F}^*\mathbf{g}^j\,, \\ \mathbf{g}^i = \mathbf{F}^{-*}\mathbf{G}^i\,, & \end{aligned} \tag{2.40}$$

such that we obtain for the push-forward of the tensor $\mathbf{E}$ to $\mathbf{e}$

$$\mathbf{e} = \mathbf{F}^{-*}(E_{ij}\mathbf{G}^i \otimes \mathbf{G}^j)\mathbf{F}^{-1} = E_{ij}\mathbf{F}^{-*}\mathbf{G}^i \otimes \underbrace{\mathbf{G}^j\mathbf{F}^{-1}}_{\mathbf{F}^{-*}\mathbf{G}^j} = E_{ij}\mathbf{g}^i \otimes \mathbf{g}^i\,. \tag{2.41}$$

Recognizing the base vector transformations **Equation** (2.22) and **Equation** (2.40) the push-forward can be easily obtained. For convective coordinates we can follow in fact a very simple rule: The components are invariant and only the base vectors are changed from $\mathbf{G}_i$ to $\mathbf{g}_i$ or $\mathbf{G}^i$ to $\mathbf{g}^i$.

2.6.2 Pull-back

Pull-back: The inverse operation to a *push-forward* is called *pull-back* and is denoted by $\mathbf{F}^{\triangleleft}(\cdots)$:

$$\mathbf{E} = \mathbf{F}^{\triangleleft}(\mathbf{e})\,, \qquad \mathbf{G}_i = \mathbf{F}^{\triangleleft}(\mathbf{g}_i)\,, \qquad \mathbf{G}^i = \mathbf{F}^{\triangleleft}(\mathbf{g}^i)\,. \tag{2.42}$$

From **Equations** (2.39), (2.42), (2.19) and (2.36) the following identities hold:

$$\mathbf{g} = \mathbf{F}_{\triangleright}(\mathbf{C})\,, \qquad \mathbf{C} = \mathbf{F}^{\triangleleft}(\mathbf{g})\,, \qquad \mathbf{b}^{-1} = \mathbf{F}_{\triangleright}(\mathbf{G})\,, \qquad \mathbf{G} = \mathbf{F}^{\triangleleft}(\mathbf{b}^{-1})\,. \tag{2.43}$$

However, although the notation is unique, as per the [Başar & Weichert 2000] the actual form of the push-forward or pull-back operation depends on the component decomposition. The relation **Equation** (2.22) implies by using **Equation** (2.27) the following tensor relation for the deformation gradient :

$$\mathbf{F} = \mathbf{g}_i \otimes \mathbf{G}^i \Rightarrow \mathbf{g}_i = \mathbf{F}\,\mathbf{G}_i = (\mathbf{g}_k \otimes \mathbf{G}^k) \cdot \mathbf{G}_i = \mathbf{g}_k \delta_i^k = \mathbf{g}_i\,. \tag{2.44}$$

Note that, if we have Kronecker-delta in one expression, we can exchange the dummy indices.

2.7 Constitutive relations in tensor notation

The Green-Lagrange strain tensor **E** and the Almansi-Euler strain tensor **e** used for the formulation of constitutive relations are expressible in terms of the following tensors:
the right Cauchy-Green tensor:

$$\mathbf{C} = \mathbf{F}^* \,\mathbf{g}\,\mathbf{F} = C_{ij}\mathbf{G}^i \otimes \mathbf{G}^j = (\mathbf{g}_i \cdot \mathbf{g}_j)\mathbf{G}^i \otimes \mathbf{G}^j\,, \tag{2.45}$$

the inverse of the left Cauchy-Green tensor:

$$\mathbf{b}^{-1} = \mathbf{F}^{-*}\,\mathbf{G}\,\mathbf{F}^{-1} = b_{ij}^{-1}\mathbf{g}^i \otimes \mathbf{g}^j = (\mathbf{G}_i \cdot \mathbf{G}_j)\mathbf{g}^i \otimes \mathbf{g}^j\,, \tag{2.46}$$

the metric tensor of the undeformed state B_0:

$$\mathbf{G} = G_{ij}\mathbf{G}^i \otimes \mathbf{G}^j\,, \tag{2.47}$$

the metric tensor of the deformed state B:

$$\mathbf{g} = g_{ij}\mathbf{g}^i \otimes \mathbf{g}^j\,, \tag{2.48}$$

where **F** is the *deformation gradient:*

$$\mathbf{F} = \mathbf{g}_i \otimes \mathbf{G}^i, \qquad \mathbf{F}^{-1} = \mathbf{G}_i \otimes \mathbf{g}^i\,. \tag{2.49}$$

We use the Green-Lagrange strain tensor to define 3D strains:

$$\begin{aligned} \mathbf{E} &= \frac{1}{2}(\mathbf{C} - \mathbf{G}) = E_{ij}\mathbf{G}^i \otimes \mathbf{G}^j = \frac{1}{2}(g_{ij} - G_{ij})\mathbf{G}^i \otimes \mathbf{G}^j\,, \\ &= \frac{1}{2}(\mathbf{g}_i \cdot \mathbf{g}_j - \mathbf{G}_i \cdot \mathbf{G}_j)\mathbf{G}^i \otimes \mathbf{G}^j\,, \end{aligned} \tag{2.50}$$

or, alternatively, the Almansi strain tensor

$$\mathbf{e} = \frac{1}{2}(\mathbf{g} - \mathbf{b}^{-1}) = e_{ij}\mathbf{g}^i \otimes \mathbf{g}^j = \frac{1}{2}(g_{ij} - G_{ij})\mathbf{g}^i \otimes \mathbf{g}^j\,. \tag{2.51}$$

Chapter 3

Shell Theory

3.1 Shell kinematics

3.1.1 Element formulation of the undeformed shell continuum

Figure 3.1 given below illustrates the undeformed shell continuum. We consider a shell element with a midsurface $\mathcal{S}_0$ and a height h, which is usually small. Each element possesses four nodes and for each node there exist six degrees of freedom (displacements and rotations). Apart from this, certain kinematical assumptions are considered. The

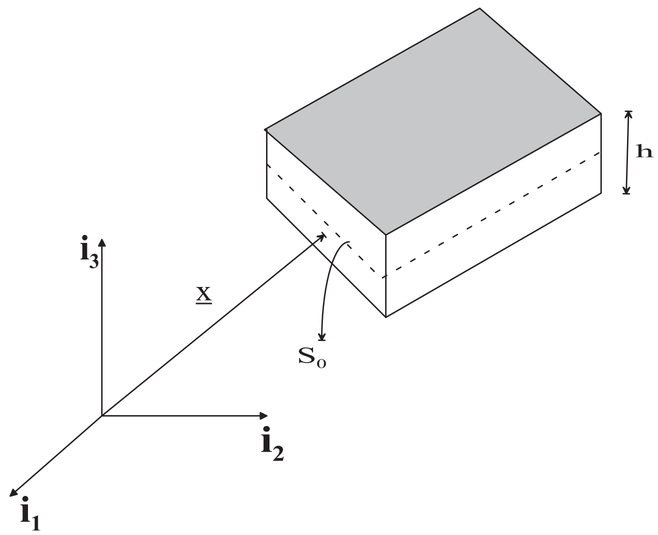

Figure **3.1**: Shell element with midsurface

kinematical assumption couples continuum-like displacements with shell variables. Here we use a so-called MINDLIN-REISSNER-kinematics which is given as follows:

$$\text{Initial state}: \quad \underbrace{\mathbf{X}}_{\text{continuum}} = \underbrace{\overset{0}{\mathbf{X}} + \theta^3 \mathbf{D}}_{\text{shell}}.$$

$$\text{Current state}: \quad \underbrace{\mathbf{x}}_{\text{continuum}} = \underbrace{\overset{0}{\mathbf{x}} + \theta^3 (\lambda \mathbf{d})}_{\text{shell}}, \qquad \theta^3 \in [-h/2, h/2].$$

3.1.2 Geometry of the deformed shell continuum

The basic idea of any shell theory is to describe the deformed shell continuum by kinematic variables referring to a reference surface selected as midsurface $\mathcal{S}_0$. Consider an arbitrary point P_0 of the shell continuum which is moved after deformation into the position P. In the present formulation the position vector $\mathbf{x} = \mathbf{x}(\theta^i)$ of P is approximated by a quadratic polynomial in thickness direction θ^3, having the form **(Figure 3.2)**.

$$\mathbf{x}(\theta^i) = \overset{0}{\mathbf{x}} + \theta^3 \overset{1}{\mathbf{x}} (\theta^\alpha) + (\theta^3)^2 \overset{2}{\mathbf{x}} (\theta^\alpha). \tag{3.1}$$

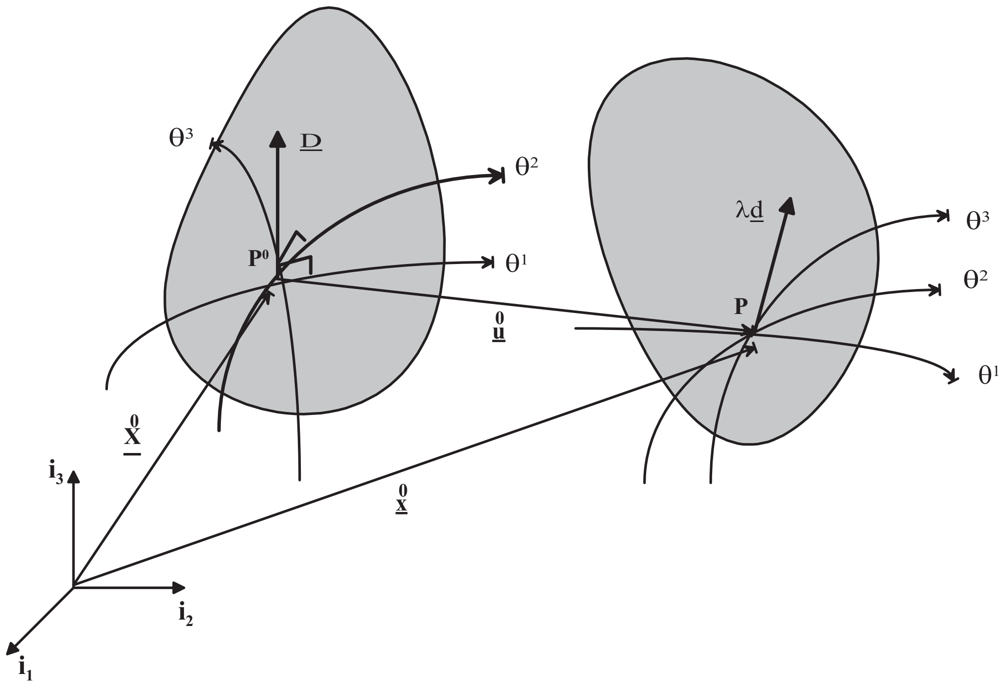

Figure **3.2**: Deformed process configuration

Herein, $\overset{0}{\mathbf{x}}$ determines the deformed midsurface $\mathcal{S}$. The higher order terms $\overset{I}{\mathbf{x}}$ ($I = 1, 2$) are shell directors which will be used in the finite-element procedure in the two forms

$$\overset{1}{\mathbf{x}} = \overset{0}{\lambda}\, d\,; \qquad \overset{2}{\mathbf{x}} = \overset{1}{\lambda}\, d\,, \tag{3.2}$$

where d is supposed to be an *inextensible* director

$$\mathbf{d} \cdot \mathbf{d} = 1 \qquad \Rightarrow \qquad \mathbf{d}_{,\alpha} \cdot \mathbf{d} = 0\,, \tag{3.3}$$

and $\overset{I}{\lambda}$ $(I = 0, 1)$ denote constant and linear stretches, respectively. The use of the kinematic model **Equation** (3.1) in combination with the multiplicative decomposition **Equation** (3.3) offers a large number of possibilities to achieve reliable finite element formulations. The multiplicative decomposition of the first order term $\overset{1}{\mathbf{x}} = \overset{0}{\lambda}\, \mathbf{d}$ permits to decouple the numerically sensitive stretch parameter $\overset{0}{\lambda}$ from the inextensible director $\mathbf{d}$ and provides thus a numerically stable finite-element procedure [Başar & Ding 1994; Simo, Fox & Rifai 1989]. But the constraint $\mathbf{d} \cdot \mathbf{d} = 1$ to be considered in this context requires a suitable parameterization of the director $\mathbf{d}$ when finite rotations are involved in the analysis. The inclusion of a higher order displacement term $\overset{2}{\mathbf{x}}$ at least in the form $\overset{2}{\mathbf{x}} = \overset{1}{\lambda}\, \mathbf{d}$ enables to simulate any cross-section wrinkling and ensures a locking-free consideration of transverse strains E_{33} [Başar & Ding 1994; Başar & Ding 1995; Sansour 1995; Verhoeven 1993]. If the kinematic model **Equation** (3.1) is truncated after the linear term, a special procedure, e.g. the *enhanced strain* formulation (EAS), has to be used in the numerical implementation to avoid the so-called Poisson-*locking* [Betsch & Stein 1995; Büchter & Ramm 1992; Büchter, Ramm & Roehl 1994].

The consideration of a higher order displacement term $\overset{2}{\mathbf{x}}$ has been proved not to be of significance concerning accuracy, as well as numerical stability. Consequently, in the present FE-formulation we neglect $\overset{2}{\mathbf{x}}$ by using, in addition, an EAS-concept to avoid locking. The displacement vector $\mathbf{u}$ of the shell continuum according to **Equation**(2.3) can be expressed as :

$$\mathbf{u} = \mathbf{x} - \mathbf{X} = (\overset{0}{\mathbf{x}} - \overset{0}{\mathbf{X}}) + \theta^3(\lambda \mathbf{d} - \mathbf{D}) = \overset{0}{\mathbf{u}} + \theta^3(\lambda \mathbf{d} - \mathbf{D})\,. \tag{3.4}$$

The displacement vector at the node i has the form:

$$\mathbf{u}^i = [\overset{0}{u}{}^i_1, \overset{0}{u}{}^i_2, \overset{0}{u}{}^i_3, \omega^i_1, \omega^i_2, \omega^i_3, \lambda^i]^T\,, \tag{3.5}$$

$\overset{0}{\mathbf{u}}$ and ω are decomposed with respect to the gobal reference frame:

$$\overset{0}{\mathbf{u}}{}^i = \overset{0}{u}{}^i_1 \mathbf{i}_1 + \overset{0}{u}{}^i_2 \mathbf{i}_2 + \overset{0}{u}{}^i_3 \mathbf{i}_3 = \overset{0}{u}{}^i_j \mathbf{i}_j, \qquad \text{(sum over j)}\,, \tag{3.6}$$

$$\boldsymbol{\omega}^i = \omega^i_1 \mathbf{i}_1 + \omega^i_2 \mathbf{i}_2 + \omega^i_3 \mathbf{i}_3 = \omega^i_j \mathbf{i}_j, \qquad \text{(sum over j)}\,. \tag{3.7}$$

The meaning of the displacement vector $\overset{0}{\mathbf{u}}{}^i$ between the midsurface position vectors $\overset{0}{\mathbf{X}}$ and $\overset{0}{\mathbf{x}}$ is clear. According to [Başar & Kintzel 2003] the directors $\mathbf{D}$ and $\lambda\mathbf{d}$ describe the deformation in thickness direction. From **Figure (3.2)** it becomes clear that the coordinate lines θ^1 and θ^2 are inscribed on the midsurface and the coordinate line θ^3 points in thickness direction. We consider unit length for the director ($\|\mathbf{D}\| = 1$) and, in fact, $\mathbf{D}$ is perpendicular to the midsurface $\mathcal{S}_0$ at the beginning. We also suppose unit length for the current director $\mathbf{d}$ ($\|\mathbf{d}\| = 1$), but using the stretching variable λ the resulting director $\lambda\mathbf{d}$ can also predict thickness stretches.

3.2 Finite rotation formulation

As being pointed out in **Section 3.1.2** the constraint $\mathbf{d}\cdot\mathbf{d} = 1$ satisfied by the inextensible shell director $\mathbf{d}$ causes difficulties in the numerical implementation if finite rotations are involved in the analysis. This is due to the nonlinearity of the constraint **Equation** (3.3) which does not provide a unique determination of $\mathbf{d}$ in the nonlinear range. As per see [Başar 1987], by considering the unit length condition, if two components of $\mathbf{d}$ are given, **Equation** (3.3) delivers two distinct solutions for the third one (by computing the square root). This difficulty can be omitted by a suitable parametrization of $\mathbf{d}$. The essential idea is to describe the rotation of $\mathbf{d}$ by such rotational variables which ensure an a priori satisfaction of the inextensibility constraint.

By considering smooth shells without intersection lines only two independent quantities are needed to describe the rotation $\mathbf{d}^k \rightarrow \mathbf{d}^{k+1}$ in each iteration step. We use a rotation vector $\Delta\boldsymbol{\omega}$ to describe the finite rotation in the form of a Taylor series expansion:

$$\mathbf{d}^{k+1} = \mathbf{d}^k + \Delta\mathbf{d} + \frac{1}{2!}\Delta^2\mathbf{d} + \frac{1}{3!}\Delta^3\mathbf{d} + \cdots \,. \tag{3.8}$$

As per see [Başar & Weichert 2000], for the finite element formulation the first and second-order variations are of special importance:

$$\delta\mathbf{d} = \delta\boldsymbol{\omega} \times \mathbf{d}, \qquad \Delta\delta\mathbf{d} = \frac{1}{2}\left(\Delta\boldsymbol{\omega} \times (\delta\boldsymbol{\omega} \times \mathbf{d}) + \delta\boldsymbol{\omega} \times (\Delta\boldsymbol{\omega} \times \mathbf{d})\right) \,. \tag{3.9}$$

3.2.1 Updated formulation

Although only two components of $\boldsymbol{\omega}$ are needed to describe a finite rotation for smooth shells we consider a rotation vector $\boldsymbol{\omega}$ with three components by preventing the singularity involved by means of a numerical procedure (see [Başar & Kintzel 2003] for details). This has the advantage that also composed shells can be considered in a unified fashion.

The so-called updated rotation described in this section has been first proposed by [Simo & Rifai 1990] and has been used in the sequel by many authors [Betsch, Gruttmann & Stein 1996; Büchter & Ramm 1992; Başar 1993]. In this case, the basic concept is to determine the actual position of the director with respect to the foregoing one by means of an *incremental* rotation vector $\Delta\boldsymbol{\omega}$ using for this purpose the following concepts: The orthogonality of the rotation tensor $\mathbf{R}$, the equivalence between a skew-symmetric tensor and its axial vector, and the definition of the Rodriguez rotation vector. Before describing this procedure, we recall that during each iteration step the first, $\delta\mathbf{d}$, and the second variation $\Delta\delta\mathbf{d}$ of the director are needed, while $\mathbf{d}$ is to be constructed once an iteration step is accomplished. Therefore we can use the rotation tensor $\Delta\boldsymbol{\omega}$ to construct the variations of the director $\mathbf{d}$. If we construct the entire series expansion of **Equation** (3.8), we see that the result is identical to a following relation:

$$\mathbf{d}^{k+1} = \mathbf{R}(\Delta\boldsymbol{\omega})\,\mathbf{d}^k\,, \tag{3.10}$$

with a rotation tensor $\mathbf{R}(\Delta\boldsymbol{\omega})$ in terms of the rotation vector $\Delta\boldsymbol{\omega}$ which describes the finite rotation of $\mathbf{d}^k$ into the new director $\mathbf{d}^{k+1}$. The explicit form of $\mathbf{R}$ is defined by:

$$\mathbf{R} = \mathbf{I} + \frac{\sin\|\Delta\boldsymbol{\omega}\|}{\|\Delta\boldsymbol{\omega}\|}\Delta\hat{\boldsymbol{\omega}} + \frac{(1-\cos\|\Delta\boldsymbol{\omega}\|)}{\|\Delta\boldsymbol{\omega}\|^2}\Delta\hat{\boldsymbol{\omega}}\Delta\hat{\boldsymbol{\omega}}\,, \tag{3.11}$$

using $\|\Delta\boldsymbol{\omega}\| = \sqrt{\Delta\boldsymbol{\omega}\cdot\Delta\boldsymbol{\omega}}$ and $\Delta\hat{\boldsymbol{\omega}}(\cdot) = \Delta\boldsymbol{\omega}\times(\cdot)$.

To get a global solution we have to solve the following equation:

$$\mathbf{K}\,\Delta\mathbf{v} \;=\; \mathbf{p}\,, \tag{3.12}$$

where $\mathbf{K}$ is the global stiffness matrix, $\Delta\mathbf{v}$ the global displacement variation and $\mathbf{p}$ the global load vector. Solving the above **Equation** (3.12) we get a solution for the unknowns $\Delta\overset{0}{\mathbf{u}}{}^i$, $\Delta\boldsymbol{\omega}^i$ and $\Delta\lambda^i$ at each node i. The new updated variables are then computed by:

$$\begin{aligned}
\overset{0}{\mathbf{x}}{}^{k+1} &= \overset{0}{\mathbf{x}}{}^{k} + \Delta\overset{0}{\mathbf{u}}\,,\\
\lambda^{k+1} &= \lambda^k + \Delta\lambda\,,\\
\mathbf{d}^{k+1} &= \mathbf{R}(\Delta\boldsymbol{\omega})\mathbf{d}^k\,,
\end{aligned} \tag{3.13}$$

at each specific node i, where the latter finite rotation is done in each iteration step. This procedure is called updated rotation formulation.

3.3 Shape functions of a bilinear element

From **Equations** (2.21), (2.23) and (2.24) we can obtain the metric tensor components g_{ij} and G_{ij}. The desired relation in terms of θ^1, θ^2 and θ^3 is obtained by using the kinematic relation **Equation** (3.1). The shape functions for the given bilinear element are defined by :

$$N^j = \frac{1}{4}(1 + \theta^{1j}\theta^1)(1 + \theta^{2j}\theta^2)\,, \tag{3.14}$$

where

$(\theta^1, \theta^2)^1 = (-1, -1)$ node 1
$(\theta^1, \theta^2)^2 = (1, -1)$ node 2
$(\theta^1, \theta^2)^3 = (1, 1)$ node 3
$(\theta^1, \theta^2)^4 = (-1, 1)$ node 4

Then we can interpolate the position vectors $\mathbf{X}$ and $\mathbf{x}$ by means of the shape functions:

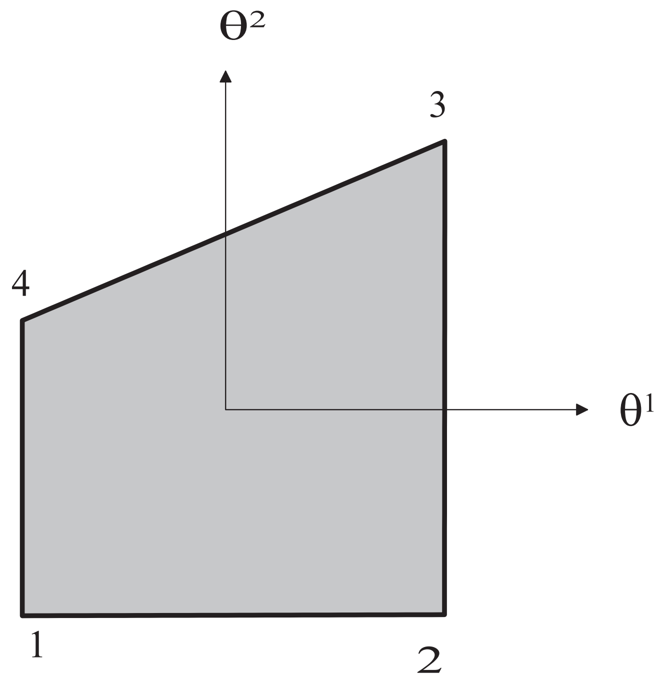

Figure **3.3**: Deformed element state

$$\mathbf{X} = \sum_{i=1}^{4} N^i(\theta^1, \theta^2)\overset{0}{\mathbf{X}}_i + \theta^3(\sum_{i=1}^{4} N^i(\theta^1, \theta^2)\mathbf{D}_i)\,, \tag{3.15}$$

$$\mathbf{x} = \sum_{i=1}^{4} N^i(\theta^1, \theta^2)\overset{0}{\mathbf{x}}_i + \theta^3(\sum_{i=1}^{4} N^i(\theta^1, \theta^2)\lambda_i)(\sum_{j=1}^{4} N^j(\theta^1, \theta^2)\mathbf{d}_j)\,. \tag{3.16}$$

From this we obtain the base vectors:

$$\mathbf{G}_1 = \frac{\partial \mathbf{X}}{\partial \theta^1} = \sum_{j=1}^{4} N^j_{,\theta^1}\overset{0}{\mathbf{X}}_j + \theta^3(\sum_{j=1}^{4} N^j_{,\theta^1}\mathbf{D}_j)\,, \tag{3.17}$$

and similarly for $\mathbf{G}_2$. $\mathbf{G}_3$ is obtained as:

$$\mathbf{G}_3 = \frac{\partial \mathbf{X}}{\partial \theta^3} = \mathbf{D} = \sum_{j=1}^{4} N^j(\theta^1, \theta^2)\mathbf{D}_j \,. \tag{3.18}$$

The current base vectors are defined by:

$$\mathbf{g}_1 = \frac{\partial \mathbf{x}}{\partial \theta^1} = \sum_{j=1}^{4} N^j_{,\theta^1} \overset{0}{\mathbf{x}}_j + \theta^3 \left((\sum_{j=1}^{4} N^j_{,\theta^1}\lambda_j)(\sum_{k=1}^{4} N^k \mathbf{d}_k) + (\sum_{j=1}^{4} N^j \lambda_j)(\sum_{k=1}^{4} N^k_{,\theta^1}\mathbf{d}_k) \right) , \tag{3.19}$$

and similarly for $\mathbf{g}_2$. $\mathbf{g}_3$ is obtained as:

$$\mathbf{g}_3 = \frac{\partial \mathbf{x}}{\partial \theta^3} = \lambda \mathbf{d} = (\sum_{j=1}^{4} N^j \lambda_j)(\sum_{k=1}^{4} N^k \mathbf{d}_k) \,. \tag{3.20}$$

For solving **Equation** (3.12) we need the first and second-order variations of the metric. This is done by variating these quantities element-wise. For example we obtain for $\mathbf{g}_3$ the following quantities:

$$\begin{aligned} \delta \mathbf{g}_3 &= \delta\lambda \mathbf{d} + \lambda \delta \mathbf{d} \\ &= (\sum_{j=1}^{4} N^j \delta\lambda_j)(\sum_{k=1}^{4} N^k \mathbf{d}_k) + (\sum_{j=1}^{4} N^j \lambda_j)(\sum_{k=1}^{4} N^k \underbrace{\delta \mathbf{d}_k}_{\delta\boldsymbol{\omega}_k \times \mathbf{d}_k}) \,, \end{aligned} \tag{3.21}$$

$$\begin{aligned} \Delta\delta \mathbf{g}_3 &= \Delta\lambda\delta\mathbf{d} + \delta\lambda\Delta\mathbf{d} + \lambda\Delta\delta\mathbf{d} \\ &= (\sum_{j=1}^{4} N^j \Delta\lambda_j)(\sum_{k=1}^{4} N^k \delta\mathbf{d}_k) + (\sum_{j=1}^{4} N^j \delta\lambda_j)(\sum_{k=1}^{4} N^k \Delta\mathbf{d}_k) \\ &\quad + (\sum_{j=1}^{4} N^j \lambda_j)(\sum_{k=1}^{4} N^k \underbrace{\Delta\delta\mathbf{d}_k}_{\frac{1}{2}(\Delta\boldsymbol{\omega}_k \times (\delta\boldsymbol{\omega}_k \times \mathbf{d}_k) + \delta\boldsymbol{\omega}_k \times (\Delta\boldsymbol{\omega}_k \times \mathbf{d}_k))}) \,, \end{aligned} \tag{3.22}$$

and similarly for $\mathbf{g}_1$ and $\mathbf{g}_2$. The variation of $\mathbf{G}_i$ is zero since the initial base vectors are constant in time.

Chapter 4

Stresses, virtual work principle and EAS-concept

As being pointed out in **Section 2** we have learned about the metric tensor components g_{ij}, G_{ij} and their significant role in continuum mechanics (see **Equation** (2.30)). Then we have seen in **Section 3** that the metric is coupled with independent displacements by means of a kinematical assumption in **Equation** (3.4). But before being able to solve **Equation** (3.12), we have to define constitutive relations.

4.1 Constitutive relations

A constitutive relation prescribes the material properties of a body. Here we use a new tensor, called stress tensor. The stress tensor is work-conjugate to the strain tensor, both are used in the principle of virtual work. As a well-known fact in linear elasticity the true CAUCHY-stresses $\boldsymbol{\sigma}$ and $\boldsymbol{\epsilon}$ are work-conjugate to each other. An often used law is called HOOKE'S law and is defined by the following linear relation.

$$\boldsymbol{\sigma} \;=\; \lambda\,(\mathrm{tr}(\boldsymbol{\epsilon}))\,\mathbf{I} + 2\mu\,\boldsymbol{\epsilon}\,, \tag{4.1}$$

where λ and μ are called LAMÉ-constants. They are coupled with YOUNG'S modulus E and POISSON'S ratio ν. These relations are given by:

$$E \;=\; \frac{\mu(3\lambda+2\mu)}{\lambda+\mu} = 2\mu(1+\nu), \qquad \nu = \frac{\lambda}{2(\lambda+\mu)}\,. \tag{4.2}$$

A strain energy function ψ^e which leads to **Equation** (4.1) can be defined in the form:

$$\psi^e(\boldsymbol{\epsilon}) \;=\; \frac{\lambda}{2}\,(\mathrm{tr}(\boldsymbol{\epsilon}))^2 + \mu\,\mathrm{tr}(\boldsymbol{\epsilon}^2)\,. \tag{4.3}$$

If **Equation** (4.1) is considered as the derivative of a potential, we get:

$$\boldsymbol{\sigma} = \frac{\partial \psi^e(\boldsymbol{\epsilon})}{\partial \boldsymbol{\epsilon}} . \tag{4.4}$$

Till this state we have considered linearity, now we want to depart from linearity in two distinct steps.

4.1.1 Second PIOLA-KIRCHHOFF stress tensor

At first, the GREEN-LAGRANGE starin tensor $\mathbf{E}$ is used instead of the nonlinear strain measure $\boldsymbol{\epsilon}$. The work-conjugate variable to $\mathbf{E}$ is the PIOLA-KIRCHHOFF stress tensor $\mathbf{S}$. Then a relation similar to **Equation** (4.1) is given by

$$\mathbf{S} = \lambda\,(\mathrm{tr}(\mathbf{G}^{-1}\mathbf{E}))\,\mathbf{G}^{-1} + 2\,\mu\,\mathbf{G}^{-1}\,\mathbf{E}\,\mathbf{G}^{-1} . \tag{4.5}$$

Similar relations to **Equations** (4.3) and (4.4) are given by:

$$\psi^e(\mathbf{E}) = \frac{\lambda}{2}\,(\mathrm{tr}(\mathbf{G}^{-1}\mathbf{E}))^2 + \mu\,\mathrm{tr}(\mathbf{G}^{-1}\mathbf{E})^2 , \tag{4.6}$$

and

$$\mathbf{S} = \frac{\partial \psi^e(\mathbf{E})}{\partial \mathbf{E}} . \tag{4.7}$$

This constitutive law is called ST. VENANT-KIRCHHOFF-law. Note that we have used the metric tensors $\mathbf{G}$ and $\mathbf{G}^{-1}$ in **Equations** (4.5) and (4.6) to emphasize that $\mathbf{S}$ has a contra-variant decomposition:

$$\mathbf{S} = S^{ij}\,\mathbf{G}_i \otimes \mathbf{G}_j , \tag{4.8}$$

and $\mathbf{S}$ can be expressed in component form as:

$$S^{ij} = \lambda\,(\mathrm{tr}(\mathbf{G}^{-1}\mathbf{E}))\,G^{ij} + 2\,\mu\,G^{ik}\,E_{kl}\,G^{lj} , \tag{4.9}$$

that means we raise the components of $\mathbf{E}$ by means of the metric G^{ij}. We want to give a short explanation, why we have written $\mathrm{tr}(\mathbf{G}^{-1}\mathbf{E})$ and not simply $\mathrm{tr}(\mathbf{E})$. If we first consider the linear strain tensor $\boldsymbol{\epsilon}$, note that we assumed $\boldsymbol{\epsilon}$ to be decomposed with respect to an orthonormal coordinate system **Equation** (2.7):

$$\boldsymbol{\epsilon} = \epsilon_{ij}\,\mathbf{i}_i \otimes \mathbf{i}_j . \tag{4.10}$$

Since the dual basis to $\mathbf{i}_i$ is the same basis ($\mathbf{i}^i = \mathbf{i}_i$) we do not distinguish between co- or contra-variant components. Thus the trace reads as:

$$\mathrm{tr}(\boldsymbol{\epsilon}) = \epsilon_{11} + \epsilon_{22} + \epsilon_{33} . \tag{4.11}$$

However, this is not the case for a curvilinear basis $\mathbf{G}_i$ or $\mathbf{g}_i$. Then we distinguish between co- and contra-variant components and we can not simply write

$$\mathrm{tr}(\mathbf{E}) \neq E_{11} + E_{22} + E_{33}\,, \tag{4.12}$$

since a trace must be an invariant. And an invariant is always computed in mixed-variant form i.e. in mixed-variant decomposition. For example:

$$\begin{aligned} &\mathrm{tr}(\mathbf{A}) = A^1_{.1} + A^2_{.2} + A^3_{.3}\,, \\ &\text{or} \\ &\mathrm{tr}(\mathbf{A}^2) = A^1_{.k}A^k_{.1} + A^2_{.l}A^l_{.2} + A^3_{.m}A^m_{.3}\,. \end{aligned} \tag{4.13}$$

4.1.2 Neo-Hooke-law

Now, we donnot only use a nonlinear strain measure but also define the constitutive law itself in non-linear form. The so-called Neo-Hooke-law is defined by:

$$\psi^e = \frac{1}{2}\,\kappa\,(\ln(J^e))^2 + \frac{1}{2}\,\mu\,[J^{e-2/3}\,\mathrm{tr}(\mathbf{G}^{-1}\mathbf{C}) - 3]\,, \tag{4.14}$$

where J^e is the square root of the third invariant of $\mathbf{C}$:

$$J^e = \sqrt{\det(\mathbf{G}^{-1}\mathbf{C})}\,, \tag{4.15}$$

and the bulk modulus κ is defined by:

$$\kappa \;=\; \frac{E}{3(1-2\nu)}\,, \qquad \mu = \frac{E}{2(1+\nu)}\,. \tag{4.16}$$

The constant quantity 3 is used in **Equation** (4.14) to get a zero value for ψ^e in the reference state. Using **Equation** (2.19), we can see that for zero strains ($\mathbf{E} = \mathbf{0}$) the right Cauchy-Green-tensor $\mathbf{C}$ is identical to the metric tensor $\mathbf{G}$:

$$\mathbf{E} \,=\, \mathbf{0} \,\Rightarrow\, \mathbf{C} \,=\, \mathbf{G}\,. \tag{4.17}$$

Then we obtain for the second term in **Equation** (4.14):

$$\frac{1}{2}\,\mu\,[\det(\mathbf{G}^{-1}\mathbf{G})^{-1/3}\,\mathrm{tr}(\mathbf{G}^{-1}\mathbf{G}) - 3]\,. \tag{4.18}$$

Since $G^{ij}G_{jk} \,=\, \delta^i_k$ is Kronecker-delta, we get :

$$\frac{1}{2}\,\mu\;[\det(\mathbf{I})^{-1/3}\,\mathrm{tr}(\mathbf{I}) - 3] \;=\; \frac{1}{2}[3-3] \;=\; 0\,, \tag{4.19}$$

if we consider **I** in matrix representation:

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.20}$$

However, we cannot use **Equation** (4.7), because ψ^e does not depend on **E** but on **C** only. By using the chain rule, it can be proved that the following relation holds:

$$\mathbf{S} = 2\frac{\partial \psi^e}{\partial \mathbf{C}}. \tag{4.21}$$

Differentiating **Equation** (4.14) with respect to **C** we arrive at the following stress tensor

$$\mathbf{S} = \kappa \ln(J^e)\,\mathbf{C}^{-1} + \mu\, J^{e\,-2/3}\left(\mathbf{G}^{-1} - \frac{1}{2}\,\mathrm{tr}(\mathbf{G}^{-1}\mathbf{C})\,\mathbf{C}^{-1}\right), \tag{4.22}$$

where $\mathbf{C}^{-1} = g^{ij}\mathbf{G}_i \otimes \mathbf{G}_j$ has been used. And for zero strains ($\mathbf{C} = \mathbf{G}\, or\, \mathbf{C}^{-1} = \mathbf{G}^{-1}$) the stress tensor vanishes. **Equation** (4.14) is just a special form of a nonlinear material law. However, for our considerations it suffices to consider only this NEO-HOOKE-law.

4.2 Energy conjugate quantities

As previously mentioned **S** and **E** are called work-conjugate because they form the stress power:

$$\mathcal{P} = \mathbf{S} : \dot{\mathbf{E}}. \tag{4.23}$$

Using the definition for **E** we obtain an alternative expression for the stress power

$$\mathcal{P} = \mathbf{S} : \frac{1}{2}\,\dot{\mathbf{C}}, \qquad \text{Since} \qquad \dot{\mathbf{E}} = \frac{1}{2}\,(\dot{\mathbf{C}} - \dot{\mathbf{G}}) = \frac{1}{2}\,\dot{\mathbf{C}}, \tag{4.24}$$

if we consider the time-independence of the second-order identity tensor $\dot{\mathbf{G}} = \dot{\mathbf{I}}$.

4.3 Second law of thermodynamics

The second law of thermodynamics in the isothermal case reads as:

$$\mathcal{D} = \mathcal{P} - \dot{\psi} \geq 0. \tag{4.25}$$

For elastic materials this relation has to be fulfilled identically that means the whole energy is stored in the material in the form of elastic lattice distortions. This distortion is reversible, since when the strains are relaxed the former structure of the lattice is recovered. That means for vanishing strains we get back the initial structure of the material, which is called reversibility.

For irreversible materials some proportions of the energy exerted on the material are dissipated and cannot be recovered. This is the case for plasticity. However, for pure reversibility the dissipated energy $\mathcal{D}$ is zero and we can state the identity in **Equation** (4.25) . If we consider, in addition, the dependence of ψ on $\mathbf{C}$ alone we get:

$$\mathcal{D} = \mathbf{S} : \frac{1}{2}\dot{\mathbf{C}} - \dot{\psi} = \mathbf{S} : \frac{1}{2}\dot{\mathbf{C}} - \frac{\partial \psi}{\partial \mathbf{C}} : \dot{\mathbf{C}} = (\mathbf{S} - 2\frac{\partial \psi}{\partial \mathbf{C}}) : \frac{1}{2}\dot{\mathbf{C}} = 0\,, \tag{4.26}$$

which states that for any strain path the hyperelastic relation **Equation** (4.21) has to be satisfied. To solve the global equation **Equation** (3.12) we need to construct the stiffness matrix and the load vector. These tensors can be obtained by using the virtual work principle.

4.4 Principle of virtual work

The principle of virtual work postulates that the external and the internal virtual work are the same:

$$\delta W^{int} = \delta W^{ext}\,, \tag{4.27}$$

that means the internal virtual work stored in the material is equal to the external virtual work done on the material by external forces. For example, if we consider contact forces $\mathbf{t}$ and body forces $\mathbf{b}$ as external forces the following relation holds :

$$W^{ext} = \int_{\Omega} \rho_0\, \mathbf{b} \cdot \mathbf{v}\, dV + \int_{\partial\Omega} \mathbf{t} \cdot \mathbf{v}\, d\mathbf{A}\,, \tag{4.28}$$

where $\partial\Omega$ is the boundary of Ω. For hyper-elasticity the inner virtual work is obtained by means of a variation of the strain energy function ψ^e:

$$W^{int} = \int_{\Omega} \psi^e\, dV\,, \tag{4.29}$$

which leads to the following formula for δW^{int}:

$$\delta W^{int} = \delta \int_{\Omega} \psi^e\, dV = \int_{\Omega} \delta\, \psi^e\, dV = \int_{\Omega} \frac{\partial \psi^e}{\partial \mathbf{C}} : \delta\, \mathbf{C} dV = \int_{\Omega} \mathbf{S} : \frac{1}{2}\, \delta\, \mathbf{C}\, dV\,. \tag{4.30}$$

Equation (4.27) represents the related stationary condition of the potential energy and demands in this form minimum of energy. To solve **Equation** (4.27) we have to employ a Newton-Raphson-procedure.

4.4.1 Linearization

At first we have to linearize this equation giving:

$$\delta W^{int} - \delta W^{ext} + \frac{\partial \delta W^{int}}{\partial \mathbf{v}} \cdot \Delta \mathbf{v} - \frac{\partial \delta W^{ext}}{\partial \mathbf{v}} \cdot \Delta\, \mathbf{v} \,=\, 0\,. \tag{4.31}$$

Since the exact solution of $\psi^e(\mathbf{C}(\mathbf{v}))$ is nonlinear, the linear approximation is a first step to the solution. To obtain the real solution we have to impose the linearized formulation **Equation** (4.31) in a number of iterative steps which finally converge to the result. At the beginning of each iteration step the linearized formulation can be given in the form of a relation of the form **Equation** (3.12), that means we have to find the quantities $\mathbf{K}$ and $\mathbf{p}$. At first, we use conservative forces that means $\mathbf{b}$ and $\mathbf{t}$ in **Equation** (4.28) are invariant during the deformation process, whereas non-conservative forces are dependent on the displacement $\mathbf{v}$. Since we use conservative forces the last term in **Equation** (4.31) vanishes identically. By equating **Equation** (4.31) with **Equation** (3.12) we can find the following relations:

$$\mathbf{K} \,=\, \frac{\partial \delta W^{int}}{\partial \mathbf{v}}, \qquad \mathbf{p} \,=\, \delta W^{ext} - \delta W^{int} \,\Rightarrow\, \mathbf{K}\,\Delta \mathbf{v} \,=\, \mathbf{p}\,. \tag{4.32}$$

In order to find the stiffness matrix $\mathbf{K}$, we proceed as follows:

$$\begin{aligned}
\frac{\partial \delta W^{int}}{\partial \mathbf{C}} : \Delta \mathbf{C} \; &= \int_\Omega \frac{1}{2}\,\delta\,\mathbf{C} : \frac{\partial \mathbf{S}}{\partial \mathbf{C}} : \Delta\,\mathbf{C} + \int_\Omega \mathbf{S} : \frac{1}{2}\,\Delta\delta\,\mathbf{C}\,dV\,, \\
&= \int_\Omega \frac{1}{2}\,\delta\,\mathbf{C} : 2\,\frac{\partial \mathbf{S}}{\partial \mathbf{C}} : \frac{1}{2}\,\Delta\,\mathbf{C} + \int_\Omega \mathbf{S} : \frac{1}{2}\,\Delta\delta\,\mathbf{C}\,dV\,, \\
&= \underbrace{\int_\Omega \frac{1}{2}\,\delta\,\mathbf{C} : \mathbb{C}^e : \frac{1}{2}\,\Delta\,\mathbf{C}}_{\text{material stiffness}} + \underbrace{\int_\Omega \mathbf{S} : \frac{1}{2}\,\Delta\delta\,\mathbf{C}\,dV}_{\text{geometric stiffness}}\;.
\end{aligned} \tag{4.33}$$

To get a relation dependent on $\Delta\mathbf{v}$ which is the variation of the displacement vector, we have to compute the variation of $\mathbf{C}$. Since

$$\delta \mathbf{C} \,=\, \delta g_{ij} \mathbf{G}^i \otimes \mathbf{G}^j + g_{ij} \delta \mathbf{G}^i \otimes \mathbf{G}^j + g_{ij} \mathbf{G}^i \otimes \delta \mathbf{G}^j\,, \tag{4.34}$$

and by virtue of the constancy of the base vectors $\mathbf{G}_i$ and $\mathbf{G}^i$ ($\delta\,\mathbf{G}^i \,=\, 0$) we finally obtain in component form:

$$\frac{\partial \delta W^{int}}{\partial \mathbf{C}} : \Delta\,\mathbf{C} \,=\, \underbrace{\int_\Omega \frac{1}{2} \delta g_{ij} \mathbb{C}^{eijkl} \frac{1}{2} \Delta g_{kl} dV}_{\text{material stiffness}} + \underbrace{\int_\Omega S^{ij} \frac{1}{2} \Delta\delta\, g_{ij} dV}_{\text{geometric stiffness}}\;. \tag{4.35}$$

The second term in **Equation** (4.35) is called geometric stiffness since it measures a change in δW^{int} solely with respect to a metric change whereas the first term measures a change with respect to a change in **S** (material law) and is called material stiffness. A short look at **Section 3.3** shows how we can arrive at the variations of the base vectors. And from these quantities we obtain the corresponding variations of the metric tensor components as given below:

$$\begin{aligned} \delta g_{ij} &= \delta \mathbf{g}_i \cdot \mathbf{g}_j + \mathbf{g}_i \cdot \delta \mathbf{g}_j , \\ \Delta\delta g_{ij} &= \Delta \mathbf{g}_i \cdot \delta \mathbf{g}_j + \delta \mathbf{g}_i \cdot \Delta \mathbf{g}_j + \Delta\delta \mathbf{g}_i \cdot \mathbf{g}_j + \mathbf{g}_i \cdot \Delta\delta \mathbf{g}_j . \end{aligned} \tag{4.36}$$

Since the variations of the base vectors are dependent on the independent kinematical quantities defined in **Section 3**, which are $\overset{0}{\mathbf{u}}$, $\boldsymbol{\omega}$ and λ, their variations are finally used to find an equation in dependence of the independent displacement variations $\Delta\mathbf{v}$. If we carry out the differentiation to obtain $2\dfrac{\partial \mathbf{S}}{\partial \mathbf{C}} = \dfrac{\partial \mathbf{S}}{\partial \mathbf{E}}$ we get for the above introduced St. Venant-Kirchhoff-material law **Equation** (4.5):

$$\mathbb{C}^e = \lambda \mathbf{G}^{-1} \otimes \mathbf{G}^{-1} + \mu(\mathbf{G}^{-1} \boxtimes \mathbf{G}^{-1} + \mathbf{G}^{-1} \times \mathbf{G}^{-1}) \tag{4.37}$$

by using tensor products $\otimes$, $\boxtimes$ and $\times$, which are defined by :

$$\begin{aligned} (\mathbf{A} \otimes \mathbf{B})^{ijkl} &= A^{ij} B^{kl} , \\ (\mathbf{A} \boxtimes \mathbf{B})^{ijkl} &= A^{ik} B^{jl} , \\ (\mathbf{A} \times \mathbf{B})^{ijkl} &= A^{il} B^{jk} , \end{aligned} \tag{4.38}$$

or in component form:

$$\mathbb{C}^{eijmn} = \lambda G^{ij} G^{mn} + \mu(G^{im} G^{jn} + G^{in} G^{jm}) . \tag{4.39}$$

4.5 Enhanced assumed strain formulation

So far we have remarked that the finite shell formulation suffers from serious deficiencies called locking. This locking phenomena have different causes and must be prevented by certain procedures.

4.5.1 Shear locking, Assumed strain formulation

Shear locking is an important locking phenomenon which occurs in thin shell structures. If we consider simple bending deformations and reduce the height of the shell, the shear modulus tends to zero with order $o(h)$ whereas the bending modulus tends to zero with order $o(h^3)$. Therfore the shear modes are overestimated for the limit $h \to 0$. The results are not the same as for a KIRCHHOFF-LOVE-type kinematical assumption which represents the solution for very thin shells. To remove shear locking we employ an assumed strain formulation [BAŞAR & KINTZEL 2003] which has been proposed by [BATHE & DVORKIN 1985]. The procedure is such: We compute the shear strains $E_{13} = E_{31}$ and $E_{23} = E_{32}$ at certain sampling points A(0,1), B(-1,0), C(0,-1) and D(1,0) and then interpolate these values bilinearly to obtain corrected shear strains according to :

$$\begin{aligned} E_{13} &= \tfrac{1}{2}(1+\theta^2)E_{13}^A + \tfrac{1}{2}(1-\theta^2)E_{13}^C, \\ E_{23} &= \tfrac{1}{2}(1+\theta^1)E_{23}^D + \tfrac{1}{2}(1-\theta^1)E_{23}^B\,. \end{aligned} \tag{4.40}$$

By doing this we obtain for the simple bending deformation case vanishing shear strains such that an overestimation cannot occur.

4.5.2 Enhanced strain formulation

In particular all other locking phenomena for bilinear elements are caused by stiffening effects due to couplings between certain strain modes. For example, in in-plane distortions the couplings between the in-plane strains E_{11}, E_{22} and $E_{12} = E_{21}$ lead a stiffening effect which can be in fact avoided by adding incompatible strain modes $\tilde{\mathbf{E}}$ to the solution. Incompatible strain mode fields are in contrast to compatible strain mode fields, which are computed from the kinematical variables, not smooth and can have jumps from element to element. Therefore they are called incompatible strain mode fields. Some of these additional strain modes are activated in the solution process and lead to a weakening effect, thus reducing locking. The above described technique is called enhanced strain formulation since we enhance the strains by adding additional strain modes to the solution. A description of this technique can be found in [BAŞAR & KINTZEL 2003]. Here we recall only the important facts. As variational basis we don't use the inner principle of virtual work, which is purely displacement oriented, but a HU-WASHIZU-principle, where the displacements, the stresses and strains can be prescribed independently. By postulating an orthogonality condition for the incompatible strain- and stress-modes the additional stress term drops out, such that in the end we obtain a principle of virtual work using solely compatible and incompatible strains in the form:

$$\delta W^{int}(\mathbf{E},\tilde{\mathbf{E}}) = \delta W^{ext}\,. \tag{4.41}$$

The process of computing $\mathbf{K}$ and $\mathbf{p}$ proceeds along similar lines with the difference that now we have also to variate with respect to the independent strain modes $\tilde{\mathbf{E}}$ ($\bar{\mathbf{E}} = \mathbf{E}+\tilde{\mathbf{E}}$).

The result is :

$$\begin{aligned}\mathbf{K} \;=\; & \int_\Omega \tfrac{1}{2}\,\delta\,\mathbf{C} : \mathbb{C}^e : \tfrac{1}{2}\,\Delta\,\mathbf{C}\,dV + \int_\Omega \delta\,\tilde{\mathbf{E}} : \mathbb{C}^e : \Delta\,\tilde{\mathbf{E}}\,dV \\ & + \int_\Omega \tfrac{1}{2}\,\delta\,\mathbf{C} : \mathbb{C}^e : \Delta\,\tilde{\mathbf{E}}\,dV + \int_\Omega \delta\,\tilde{\mathbf{E}} : \mathbb{C}^e : \tfrac{1}{2}\,\Delta\,\mathbf{C}\,dV\,,\end{aligned} \tag{4.42}$$

plus the geometric stiffness term (see **Equation** (4.35)) and

$$\mathbf{p} \;=\; \delta\,W^{ext} - \int_\Omega \mathbf{S} : \frac{1}{2}\,\delta\,\mathbf{C}\,dV - \int_\Omega \mathbf{S} : \delta\,\tilde{\mathbf{E}}\,dV\,. \tag{4.43}$$

A further consequence is that the deformation gradient $\mathbf{F}$ now has the following polar decomposition:

$$\bar{\mathbf{F}} \;=\; \mathbf{R}\bar{\mathbf{U}} \;=\; \bar{\mathbf{v}}\mathbf{R}\,, \tag{4.44}$$

where $\bar{\mathbf{U}}$ represents the sum of compatible (resulting in $\mathbf{U}$ or $\mathbf{v}$) and incompatible stretching modes. Since we are adding $\tilde{\mathbf{E}}$ on element level we are able to condensate the additional unknowns, which we call α_i ($i \;=\; 1, 2, \cdots, 11$), on element level. Thus we obtain the typical form of **Equation** (3.12) solely in dependance of the compatible displacement variation $\Delta\mathbf{v}$:

$$\tilde{\mathbf{K}}_T\Delta\mathbf{v} \;=\; \tilde{\mathbf{p}}\,, \tag{4.45}$$

where $\tilde{\mathbf{K}}_T$ and $\tilde{\mathbf{p}}$ are defined according to [Başar & Kintzel 2003]. After solving **Equation** (4.45) we can compute $\mathbf{v}^{k+1}$ and with the solution for $\Delta\mathbf{v}$ we can update the additional unknown vectorc $\alpha^k \;\rightarrow\; \alpha^{k+1}$ which is like the director updated in each iteration step. Since $\tilde{\mathbf{E}}$ is added to the strains but not to the displacements $\mathbf{v}$ the deformation gradient $\mathbf{F}$ does contain only compatible modes in **Equations** (2.15) and (2.16). However, if we need $\bar{\mathbf{F}}$ also including incompatible stretches we have to employ a backward substitution process explained in the following:

- At first we compute the stretch tensor tensor $\mathbf{U}$ from the compatible right Cauchy-Green tensor $\mathbf{C}$ using principal stretches λ_A^2 by solving the characteristic polynomial in the following closed form ([Simo & Hughes 1998]):

  Let I_A, ($A \;=\; 1, 2, 3$) be the principal invariants of $\mathbf{C}$, defined as

$$\begin{aligned} I_1 \;&=\; \mathrm{tr}(\mathbf{G}^{-1}\mathbf{C}), \\ I_2 \;&=\; \frac{1}{2}(I_1^2 - \mathrm{tr}(\mathbf{G}^{-1}\mathbf{C}\mathbf{G}^{-1}\mathbf{C})), \\ I_3 \;&=\; \det(\mathbf{G}^{-1}\mathbf{C}), \end{aligned} \tag{4.46}$$

then

$$
\begin{aligned}
b &= I_2 - \frac{I_1^2}{3}\,,\\
c &= -\frac{2}{27}\,I_1^3 + \frac{I_1\,I_2}{3} - I_3\,,\\
&\text{If}\,(|b| \,\leq\, \text{tol})\,\text{then}:\\
x_A &= -c^{1/3}\,,\\
&\text{or else}\\
m &= 2\sqrt{\frac{-b}{3}}\,,\\
n &= \frac{3\,c}{m\,b}\,,\\
&\text{If}(n > 1) \quad \text{(rounding error)} \quad \text{then} \quad n = 0.9999999999999999\\
t &= \frac{\arctan\left[\sqrt{1-n^2}/n\right]}{3}\,,\\
x_A &= m\cos\left[t + 2(A-1)\frac{\pi}{3}\right]\,,\\
&\text{endif}\\
\lambda_A^2 &= x_A + \frac{I_1}{3}\,.
\end{aligned}
\tag{4.47}
$$

- Then we compute the stretch tensor $\mathbf{U}$

$$
\mathbf{U} = \frac{1}{I_U II_U - III_U}\left[-\mathbf{G}^{-1}\mathbf{C}\mathbf{G}^{-1}\mathbf{C} + (I_U^2 - II_U)\mathbf{G}^{-1}\mathbf{C} + I_U III_U \mathbf{I}\right]\,, \tag{4.48}
$$

where the invariants of $\mathbf{U}$ are given by

$$
\begin{aligned}
I_U &= \lambda_1 + \lambda_2 + \lambda_3\,,\\
II_U &= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3\,,\\
III_U &= \lambda_1\lambda_2\lambda_3\,,
\end{aligned}
\tag{4.49}
$$

- Using the deformation gradient $\mathbf{F}$, which is known from the compatible displacements $\mathbf{v}$ we obtain the unique rotation tensor $\mathbf{R}$ by computing

$$
\mathbf{R} = \mathbf{F}\mathbf{U}^{-1}\,. \tag{4.50}
$$

- Now we derive the stretch tensor $\bar{\mathbf{U}}$ from the right Cauchy-Green tensor $\mathbf{C} + \tilde{\mathbf{C}}$ which now includes both compatible and incompatible stretches, using again

Equations (4.46), (4.47), (4.48) and considering the corresponding invariants from **Equation** (4.49) which belong to $\bar{\mathbf{U}}$ by using the same metric tensor $\mathbf{G}$ in both calculations.

- Finally determination of the new deformation gradient $\bar{\mathbf{F}} = \mathbf{R}\,\bar{\mathbf{U}} = \mathbf{F}\,\mathbf{U}^{-1}\,\bar{\mathbf{U}}$.

Chapter 5

Plasticity

5.1 Introduction

Plastic deformations in metals are characterized by a movement of dislocations along local glide planes within the lattice structure. During this gliding process atoms move to other positions away from their neighbouring atoms (recombination) in the atomic structure, but the lattice structure itself is not altered. Therefore we say that there is no memory effect present in plasticity in contrast to elasticity where the lattice structure is elastically deformed and distorted. The main problem is to transport the events on the microscale to the macroscale. For this purpose there are two main approaches. One approach is to describe the dislocation movements, the debonding and rebonding of neighbouring atoms in mathematical terms and to transport these characteristics one or more scales upwards. Then the material behaviour on the macroscale is in fact dependent on atomic movements and debonding processes. This approach is called homogenisation, the complex heterogeneous structure is like in integral calculus made smoother or more homogeneous by going a scale upwards. Another approach is to describe the material behavior directly on the macroscale. This can be done by disregarding the actual process which occurs in the material. This approach is called phenomenological.

5.2 J_2-plasticity

To describe plastic deformations phenomenologically we have at first to figure out the main characteristics of plastic deformations on the macroscale. One characteristic is the propensity to glide along slipbands (now on a greater scale), if the material is subjected to shearing. In contrast, if we compress or expand a metal by subjecting it to a hydrostatical loading for example, i.e. if we push or pull in all directions, nothing happens. Another characteristic of the material is yielding above a certain yield limit. This limit is not invariant (like in ideal plasticity), but can grow with increasing deformation. The latter process is called hardening. Now we want to express these characteristics in mathematical terms. It is advantageous to consider, first, CAUCHY stresses in principal axes. Since plasticity is independent on a hydrostatical loading, the mathematical relation has not to depend on the trace of the stress tensor

$$\mathrm{tr}(\boldsymbol{\sigma}) \;=\; \sigma_1 + \sigma_2 + \sigma_3\,. \tag{5.1}$$

However, it should depend on shear stresses, which are given by

$$\frac{(\sigma_1 - \sigma_2)}{2}\,, \qquad \frac{(\sigma_1 - \sigma_3)}{2}\,, \qquad \frac{(\sigma_2 - \sigma_3)}{2}\,. \tag{5.2}$$

If we compute the deviatoric stresses in the form

$$\mathrm{dev}(\boldsymbol{\sigma}) \;=\; \boldsymbol{\sigma} - \frac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})\mathbf{i} \tag{5.3}$$

by substracting the trace, and evaluate the J_2-invariant, we get:

$$\sqrt{\frac{3}{2}(\boldsymbol{\sigma} - \frac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})\mathbf{i}) : (\boldsymbol{\sigma} - \frac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})\mathbf{i})} \;=\; \sqrt{\frac{(\sigma_1 - \sigma_2)^2}{2} + \frac{(\sigma_1 - \sigma_3)^2}{2} + \frac{(\sigma_2 - \sigma_3)^2}{2}} \tag{5.4}$$

We can see that the above term represents a weighted some of the shear stresses which are used in **Equation** (5.2), where initially no loading direction is preferred. Now we want to relate this quantity to the yield limit in the uniaxial stress case. Then the stress tensor is given by:

$$\boldsymbol{\sigma} \;=\; \begin{pmatrix} \sigma_{Y_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{5.5}$$

and evaluating the second term in **Equation** (5.4) we simply get σ_{Y_0}. This means, a condition to be fulfilled at the point of yielding is:

$$\sqrt{\frac{3}{2}\underbrace{\mathrm{dev}(\boldsymbol{\sigma}) : \mathrm{dev}(\boldsymbol{\sigma})}_{J_2-\text{invariant}}} - \sigma_{Y_0} \;=\; 0\,, \tag{5.6}$$

or written in another form:

$$F \;=\; \sqrt{\mathrm{dev}(\boldsymbol{\sigma}) : \mathrm{dev}(\boldsymbol{\sigma})} - \sqrt{\frac{2}{3}}\sigma_{Y_0} \;=\; 0\,. \tag{5.7}$$

Before reaching the yield limit the left-hand side term in F is obviously smaller than σ_{Y_0} therefore for elastic behaviour $F < 0$ is satisfied. In this case F can be used as a kind of switch:

$$F < 0 \rightarrow \text{elastic behaviour}, \qquad F = 0 \rightarrow \text{plastic behaviour}\,, \tag{5.8}$$

and therefore F is called yield condition. Note that the condition $F > 0$ makes no sense, since the J_2-invariant can not exceed the yield limit. However, if the yield limit is not fixed, but grows with increasing deformations, we may substitute σ_{Y_0} by a variable term $q(\alpha)$ in terms of the equivalent plastic strain α, which gives:

$$F = \sqrt{\text{dev}(\boldsymbol{\sigma}) : \text{dev}(\boldsymbol{\sigma})} - \sqrt{\frac{2}{3}} q(\alpha)\,, \tag{5.9}$$

where $q = \sigma_{Y_0} + H\alpha$ is the sum of the initial yield limit σ_{Y_0} and a term α representing a strain-like variable where H is the hardening modulus. Now we turn to a form of the yield function, which will be used in the finite strain regime. It is given by :

$$F = \sqrt{\text{dev}(\mathbf{g}\boldsymbol{\tau})\text{dev}(\mathbf{g}\boldsymbol{\tau}) : \mathbf{i}} - \sqrt{\frac{2}{3}} q(\alpha) \leq 0\,. \tag{5.10}$$

Here we have used KIRCHHOFF-stresses $\boldsymbol{\tau}$ which are defined in contra-variant decomposition:

$$\boldsymbol{\tau} = \tau^{ij} \mathbf{g}_i \otimes \mathbf{g}_j\,, \tag{5.11}$$

$\boldsymbol{\tau}$ is coupled with the second PIOLA-KIRCHHOFF-stress tensor $\mathbf{S}$ by means of push-forward-relation:

$$\mathbf{S} = \mathbf{F}^{\triangleleft}(\boldsymbol{\tau}), \qquad \boldsymbol{\tau} = \mathbf{F}_{\triangleright}(\mathbf{S})\,, \tag{5.12}$$

i.e. the components are, by using convective coordinates, like in **Equation** (2.34) ($E_{ij} = e_{ij}$) the same :

$$\tau^{ij} = S^{ij}\,. \tag{5.13}$$

Here we have used the spatial metric tensor $\mathbf{g}$ in F, in contrast to $\mathbf{G}$ which had been employed to transform $\mathbf{S}$ and $\mathbf{E}$ in **Equation** (4.5) in the context of elasticity, to lower the components of $\boldsymbol{\tau}$. In tensor notation the deviatoric part of $\boldsymbol{\tau}$ then reads as :

$$\text{dev}(\mathbf{g}\boldsymbol{\tau}) = \mathbf{g}(\boldsymbol{\tau} - \frac{1}{3}\text{tr}(\mathbf{g}\boldsymbol{\tau})\mathbf{g}^{-1}) = \mathbf{g}\boldsymbol{\tau} - \frac{1}{3}\text{tr}(\mathbf{g}\boldsymbol{\tau})\mathbf{i}\,. \tag{5.14}$$

5.2.1 Plastic deformation measure

For $F < 0$ the elastic behaviour is described by an elastic law which has to be satisfied for a tiny amount below $F = 0$ and still has to be fulfilled in the limit for F tending to zero.

But, in the plastic regime ($F = 0$), $\boldsymbol{\tau}$ has to be either fixed with increasing deformations like in ideal plasticity or can increase by a certain amount due to hardening. Therefore the total deformations which are increasing anyway can not be purely elastic, since to obtain a steady stress tensor the elastic deformations have to be also fixed. Therefore we have to introduce a plastic deformation measure. In the linear theory an additive relationship is typically used:

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^e + \boldsymbol{\epsilon}^p \,, \tag{5.15}$$

where $\boldsymbol{\epsilon}^e$ are the elastic and $\boldsymbol{\epsilon}^p$ are the plastic linear strains. Whereas the total strains $\boldsymbol{\epsilon}$ are computed from the total displacements according to **Equation** (2.7) the elastic strains and plastic strains are somehow artificial and are not computed from any kind of displacement. Thus we call $\boldsymbol{\epsilon}^p$ (also $\boldsymbol{\epsilon}^e$) an internal variable since we cannot derive these values by pure experimental observations like an external variable, e.g. $\boldsymbol{\epsilon}$, which can be derived from the total displacements $\mathbf{u}$, which in turn, can be measured at the specimen. In the finite strain regime we employ instead of the above introduced additive relationship **Equation** (5.15) a multiplicative decomposition of the deformation gradient into an elastic and plastic part

$$\mathbf{F} = \mathbf{F}^e \mathbf{F}^p \,. \tag{5.16}$$

This postulate can be motivated as follows : If we have pure plasticity the deformation gradient is given by $\mathbf{F} = \mathbf{F}^p$ which is purely plastic. Now, if we depart from this state and have elastic deformations, the elastic deformation gradient is added to the current deformation gradient $\mathbf{F}^p$ resulting in $\mathbf{F} = \mathbf{F}^e\mathbf{F}^p$. If we now relax the material we arrive again at the initial state which had been $\mathbf{F} = \mathbf{F}^p$.

5.2.2 Plasticity formulated in the intermediate configuration

If we consider the plastic or elastic deformation gradient like in **Equation** (2.22) or **Equation** (2.40) we see in fact that by the above decomposition **Equation** (5.16) we introduce two further base vectors :

$$\hat{\mathbf{G}}_i = \mathbf{F}^p \mathbf{G}_i = \mathbf{F}^{e\,-1} \mathbf{g}_i \,, \qquad \hat{\mathbf{G}}^i = \mathbf{F}^{p\,-T} \mathbf{G}^i = \mathbf{F}^{e\,*} \mathbf{g}^i \,, \tag{5.17}$$

along with the following metric tensors

$$\hat{\mathbf{G}} = \hat{G}_{ij} \hat{\mathbf{G}}^i \otimes \hat{\mathbf{G}}^j \,, \qquad \hat{\mathbf{G}}^{-1} = \hat{G}^{ij} \hat{\mathbf{G}}_i \otimes \hat{\mathbf{G}}_j \,. \tag{5.18}$$

These tensors define a so-called intermediate configuration, which is intermediate between the reference and current configuration. This configuration is stress-free, since the elastic stresses are relaxed, and incompatible. Since there is no displacement field from which we can derive $\mathbf{F}^p$, i.e. no smooth and compatible field, it is called incompatible.

Now we make the following assumption: Elastic strains are solely defined by means of $\mathbf{F}^e$ such that similar to **Equation** (2.20) and **Equation** (2.30) we can get the following relation:

$$\hat{\mathbf{E}} = \frac{1}{2}(\hat{\mathbf{C}} - \hat{\mathbf{G}}) = \frac{1}{2}(\mathbf{F}^{e\,*}\,\mathbf{g}\,\mathbf{F}^e - \hat{\mathbf{G}}) = \frac{1}{2}(g_{ij} - \hat{G}_{ij})\hat{\mathbf{G}}^i \otimes \hat{\mathbf{G}}^j\,, \tag{5.19}$$

where $\hat{\mathbf{C}}$ is the so-called elastic right-Cauchy-Green-tensor defined in component form by $\hat{\mathbf{C}}_{ij} = g_{ij}$ that means, we substract the intermediate metric $\hat{G}_{ij}$ from the spatial metric g_{ij}. For pure elasticity we can obtain the elastic stress tensor as given below

$$\hat{\mathbf{S}} = \frac{\partial \psi^e(\hat{\mathbf{E}})}{\partial \hat{\mathbf{E}}}\,, \tag{5.20}$$

which is just the pull-back of $\boldsymbol{\tau}$ or the push-forward of $\mathbf{S}$:

$$\hat{\mathbf{S}} = \mathbf{F}^{e\,\triangleleft}(\boldsymbol{\tau})\,, \qquad \hat{\mathbf{S}} = \mathbf{F}^p_{\triangleright}(\mathbf{S})\,. \tag{5.21}$$

The components : $\hat{S^{ij}} = S^{ij} = \tau^{ij}$ are for convective coordinates the same like in **Equation** (5.13).

5.2.3 Strain energy in the intermediate configuration

Using the intermediate metric we may define a St.-Venant-Kirchhoff-hyperelastic law:

$$\psi^e(\hat{\mathbf{E}}) = \frac{\lambda}{2}\,(\mathrm{tr}(\hat{\mathbf{G}}^{-1}\hat{\mathbf{E}}))^2 + \mu\,\mathrm{tr}(\hat{\mathbf{G}}^{-1}\hat{\mathbf{E}})^2\,, \tag{5.22}$$

or a Neo-Hooke-law in the form:

$$\psi^e = \frac{1}{2}\,\kappa\,(\ln(J^e))^2 + \frac{1}{2}\,\mu\,[J^{e-2/3}\,\mathrm{tr}(\hat{\mathbf{G}}^{-1}\hat{\mathbf{C}}) - 3]\,, \tag{5.23}$$

with

$$J^e = \sqrt{\det(\hat{\mathbf{G}}^{-1}\hat{\mathbf{C}})}\,, \tag{5.24}$$

where $\hat{\mathbf{C}} = \mathbf{F}^{e\,\triangleleft}(\mathbf{g}) = \mathbf{F}^{e\,*}\,\mathbf{g}\,\mathbf{F}^e$. We can also use the following law instead of **Equation** (5.20)

$$\hat{\mathbf{S}} = 2\,\frac{\partial \psi^e(\hat{\mathbf{C}})}{\partial \hat{\mathbf{C}}}\,. \tag{5.25}$$

Besides the elastic strain energy function ψ^e which defines the reversible amount of energy stored in the material, a plastic part ψ^p can be introduced, which defines the irreversible part of energy stored in the material due to hardening, as follows:

$$\psi^p = \sigma_{Y_0}\alpha + \frac{1}{2}H\alpha^2\,, \tag{5.26}$$

such that

$$\psi \;=\; \psi^e + \psi^p \,. \tag{5.27}$$

Similar to **Equation** (2.43) the following relations holds:

$$\mathbf{g} \;=\; \mathbf{F}^e_{\triangleright}(\hat{\mathbf{C}})\,, \quad \hat{\mathbf{C}} \;=\; \mathbf{F}^{e\,\triangleleft}(\mathbf{g})\,, \quad \mathbf{b}^e \;=\; \mathbf{F}^e_{\triangleright}(\hat{\mathbf{G}}^{-1})\,, \quad \hat{\mathbf{G}}^{-1} \;=\; \mathbf{F}^{e\,\triangleleft}(\mathbf{b}^e)\,, \tag{5.28}$$

where the elastic left CAUCHY-GREEN-tensor $\mathbf{b}^e \;=\; \mathbf{F}^e\,\hat{\mathbf{G}}^{-1}\,\mathbf{F}^{e\,*}$ has been introduced. By virtue of **Equation** (5.28) any tensor in **Equation** (5.23) can be pushed forward and we end up with the following form of the constitutive law:

$$\psi^e \;=\; \frac{1}{2}\,\kappa\,(\ln(J^e))^2 + \frac{1}{2}\,\mu\,[J^{e-2/3}\mathrm{tr}(\mathbf{b}^e\mathbf{g}) - 3]\,, \tag{5.29}$$

with

$$J^e \;=\; \sqrt{\det(\mathbf{b}^e\mathbf{g})}\,. \tag{5.30}$$

To put it short, any invariant (trace, determinant) in terms of $(\hat{\mathbf{G}}^{-1}\hat{\mathbf{C}})$ is identical to an invariant in terms of $(\mathbf{b}^e\mathbf{g})$. The similarity of both expressions becomes clear if we consider the component relation of the trace $\mathrm{tr}(\mathbf{b}^e\mathbf{g})$ holding for convective coordinates:

$$\mathrm{tr}(\hat{\mathbf{G}}^{-1}\hat{\mathbf{C}}) \;=\; \hat{G}^{ij}g_{ij} \;=\; \mathrm{tr}(\mathbf{b}^e\mathbf{g})\,, \tag{5.31}$$

that means in convective coordinates the components of $\hat{\mathbf{C}}$ are simply g_{ij} and those of $\mathbf{b}^e$ are $\hat{G}^{ij}$ which obviously proves the coincidence of both relations **Equation** (5.23) and **Equation** (5.29). In the literature [BAŞAR & ECKSTEIN 1997] the metric tensor $\mathbf{g}$ is typically left out and **Equation** (5.30) is simply written as:

$$J^e \;=\; \sqrt{\det(\mathbf{b}^e)}\,. \tag{5.32}$$

However, since in fact the spatial metric $\mathbf{g}$ has to be used to form the invariant J^e, we make this dependence on $\mathbf{g}$ explicit.

5.3 Evolution equations

To find the constitutive relations we have to consider the local dissipation inequality, which is similar to **Equation** (4.26):

$$\begin{aligned}
\mathcal{D} \;&= \mathbf{S} : \frac{1}{2}\dot{\mathbf{C}} - \dot{\psi} \;\geq\; 0\,, \\
&= \mathbf{F}^p_{\triangleright}(\mathbf{S}) : \frac{1}{2}\mathbf{F}^p_{\triangleright}(\dot{\mathbf{C}}) - \frac{\partial\psi}{\partial\hat{\mathbf{C}}} : \dot{\hat{\mathbf{C}}} - \frac{\partial\psi}{\partial\hat{\mathbf{G}}^{-1}} : \dot{\hat{\mathbf{G}}}^{-1} - \frac{\partial\psi}{\partial\alpha}\dot{\alpha} \;\geq\; 0\,, \\
&= \hat{\mathbf{S}} : \frac{1}{2}\mathbf{F}^p_{\triangleright}(\dot{\mathbf{C}}) - \frac{\partial\psi}{\partial\hat{\mathbf{C}}} : \dot{\hat{\mathbf{C}}} - q\dot{\alpha} \;\geq\; 0\,.
\end{aligned} \tag{5.33}$$

Here we have assumed not equality (=0) but inequality (≥ 0) since some energy is dissipated and not stored in ψ. $\hat{\mathbf{G}}^{-1}$ represents an identity tensor and the time derivative of an identity tensor is zero. Also we have considered

$$q = \frac{\partial \psi}{\partial \alpha} . \tag{5.34}$$

from **Equation** (5.26) which delivers

$$q = \sigma_{Y_0} + H\alpha . \tag{5.35}$$

We use a so-called LIE-derivative of $\hat{\mathbf{C}}$ [BAŞAR & WEICHERT 2000] which is defined by:

$$\mathrm{L}_{\mathbf{v}_p}(\hat{\mathbf{C}}) = \mathbf{F}^p_{\triangleright} \, \dot{\overline{(\mathbf{F}^{p\,\triangleleft}(\hat{\mathbf{C}}))}} . \tag{5.36}$$

Using $\mathbf{C} = \mathbf{F}^{\triangleleft}(\mathbf{g}) = \mathbf{F}^{p\,\triangleleft}(\hat{\mathbf{C}})$, this is identical to

$$\mathrm{L}_{\mathbf{v}^p}(\hat{\mathbf{C}}) = \mathbf{F}^p_{\triangleright}(\dot{\mathbf{C}}) = \dot{\hat{\mathbf{C}}} + \hat{\mathbf{L}}^{p\,*}\hat{\mathbf{C}} + \hat{\mathbf{C}}\hat{\mathbf{L}}^p \Rightarrow \dot{\hat{\mathbf{C}}} = \mathbf{F}^p_{\triangleright}(\dot{\mathbf{C}}) - \hat{\mathbf{L}}^{p\,T}\hat{\mathbf{C}} - \hat{\mathbf{C}}\hat{\mathbf{L}}^p , \tag{5.37}$$

where $\hat{\mathbf{L}}^p = \dot{\mathbf{F}}^p\mathbf{F}^{p\,-1}$ is the plastic velocity gradient. Putting **Equation** (5.37) into the dissipation inequality we get:

$$\mathcal{D} = \hat{\mathbf{S}} : \frac{1}{2}\,\mathbf{F}^p_{\triangleright}(\dot{\mathbf{C}}) - \frac{\partial \psi}{\partial \hat{\mathbf{C}}} : (\mathbf{F}^p_{\triangleright}(\dot{\mathbf{C}}) - \hat{\mathbf{L}}^{p\,*}\hat{\mathbf{C}} - \hat{\mathbf{C}}\hat{\mathbf{L}}^p) - q\dot{\alpha} \geq 0 , \tag{5.38}$$

which must hold for any possible tensor $\mathbf{F}^p_{\triangleright}(\dot{\mathbf{C}})$ and therefore delivers the elastic law

$$\hat{\mathbf{S}} = 2\,\frac{\partial \psi}{\partial \hat{\mathbf{C}}} , \tag{5.39}$$

and a reduced dissipation inequality

$$\frac{\partial \psi}{\partial \hat{\mathbf{C}}} : (\hat{\mathbf{L}}^{p\,*}\hat{\mathbf{C}} + \hat{\mathbf{C}}\hat{\mathbf{L}}^p) - q\dot{\alpha} \geq 0 , \tag{5.40}$$

which can be transformed into

$$2\,\hat{\mathbf{C}}\,\frac{\partial \psi}{\partial \hat{\mathbf{C}}} : \hat{\mathbf{L}}^p - q\dot{\alpha} \geq 0 \tag{5.41}$$

Comparing with **Equation** (5.39) we finally find the following expression for the dissipation inequality:

$$\hat{\mathbf{C}}\,\hat{\mathbf{S}} : \hat{\mathbf{L}}^p - q\dot{\alpha} \geq 0 , \tag{5.42}$$

which by a push-forward with $\mathbf{F}^e(\mathbf{g} = \mathbf{F}^e_{\triangleright}(\hat{\mathbf{C}}), \boldsymbol{\tau} = \mathbf{F}^e_{\triangleright}(\hat{\mathbf{S}}), \mathbf{l}^p = \mathbf{F}^e_{\triangleright}(\hat{\mathbf{L}}^p))$ leads to

$$\mathbf{g}\boldsymbol{\tau} : \mathbf{l}^p - q\dot{\alpha} \geq 0 . \tag{5.43}$$

We see that the tensor $\mathbf{g}\boldsymbol{\tau}$ and the scalar q are variables which also appear in the yield function F. As next step we form the so-called Kuhn-Tucker condition $\dot{\gamma}F$, where we have introduced a strain-like scalar $\dot{\gamma}$ which is called consistency-parameter, and add this value to the dissipation inequality. If the current material behaviour is elastic ($F < 0$) the consistency parameter is zero ($\dot{\gamma} = 0$). For plastic material behaviour ($F = 0$) the consistency parameter takes a certain value but is always positive ($\dot{\gamma} > 0$). Therefore we see, that the term $\dot{\gamma}F$ is always zero. It doesn't matter if we have elastic or plastic material behaviour, that means we can set :

$$\mathbf{g}\boldsymbol{\tau} : \mathbf{l}^p - q\dot{\alpha} - \dot{\gamma}F \geq 0\,. \tag{5.44}$$

Now we know that there is some energy dissipated. Since the laws of physics are somehow always extremum principles we could be right to demand maximum of dissipated energy also. By doing this, we have a nice possibility to derive the evolution laws for $\mathbf{l}^p$ and α simply by demanding a maximum. By recalling the familiar laws of differential calculus we know that the first derivative has to be zero. Therefore, if we differentiate this equation firstly by $\mathbf{g}\boldsymbol{\tau}$ and secondly by q then we obtain the following two distinct results:

$$\mathbf{l}^p = \dot{\gamma}\frac{\partial F}{\partial \mathbf{g}\boldsymbol{\tau}} = \dot{\gamma}\,\mathbf{n}, \qquad \dot{\alpha} = -\dot{\gamma}\frac{\partial F}{\partial q}\,, \tag{5.45}$$

i.e. we obtain so-called normality rules, where $\mathbf{n}$ is the normal to the yield surface, which is always perpendicular to F at the current point of the yield surface. Using **Equation** (5.45) and integrating we obtain the following evolution laws:

$$\begin{aligned} \mathbf{F}^p_{(n+1)} &= \exp(\Delta\,\gamma\,\hat{\mathbf{N}}_{(n+1)})\,\mathbf{F}^p_{(n)}\,, \\ \alpha_{(n+1)} &= \alpha_{(n)} + \sqrt{\frac{2}{3}}\,\Delta\,\gamma\,, \end{aligned} \tag{5.46}$$

where we have used the pull-back of $\mathbf{n}$ in the form $\hat{\mathbf{N}} = \mathbf{F}^{e\,\triangleleft}(\mathbf{n})$. The above law **Equation** (5.46.1) is an exponential function which is the exact solution to the following kind of evolution law:

$$\hat{\mathbf{L}}^p = \dot{\mathbf{F}}^p\mathbf{F}^{p\,-1} = \dot{\gamma}\hat{\mathbf{N}} \Rightarrow \mathbf{F}^p_{(n+1)} = \exp(\Delta\gamma\,\hat{\mathbf{N}}_{(n+1)})\mathbf{F}^p_{(n)}\,, \tag{5.47}$$

where $\hat{\mathbf{L}}^p$ is the plastic velocity gradient.

5.4 Isotropic elasto-plasticity

The model developed so far allowed the consideration of anisotropic elastic material behaviour. However, then all nine components of the plastic deformation gradient have to be stored. If we allow only isotropic elastic behaviour it is also possible to consider an evolution of the elastic left CAUCHY-GREEN tensor which has only six components if we consider its symmetry. Thus, instead of describing the material model in the intermediate configuration, we will develop the material model in the current configuration once more for direct comparison. Since the main relations are just the push forward of those of the intermediate configuration the component relations of the constitutive laws are the same. We will give the component expressions of the constitutive relations shortly. Now we need not only the differentiations of those laws with respect to the elastic right Cauchy-Green tensor but also with respect to the elastic left Cauchy-Green tensor. Thus the corresponding relations are given.

5.4.1 Constitutive laws

$$\psi_e = \frac{1}{2}\,\kappa\,(lnJ^e)^2 + \frac{1}{2}\,\mu\,(J^{e\,-2/3}\,tr(\mathbf{g}\mathbf{b}^e) - 3)$$

$$\boldsymbol{\tau} \;= 2\frac{\partial \psi^e}{\partial \mathbf{g}} = \kappa\,lnJ^e\,\mathbf{g}^{-1} + \mu\,J^{e\,-2/3}\,(\mathbf{b}^e - \frac{1}{3}\,I_{\mathbf{b}^e}\,\mathbf{g}^{-1})$$

$$\mathbf{g}\boldsymbol{\tau} \;= \kappa\,lnJ^e\,\mathbf{i} + \mu\,J^{e\,-2/3}\,(\mathbf{g}\mathbf{b}^e - \frac{1}{3}\,I_{\mathbf{b}^e}\,\mathbf{i})$$

$$\frac{\partial \boldsymbol{\tau}}{\partial \mathbf{b}^e} \;= \frac{1}{2}\,\kappa\,\mathbf{g}^{-1}\times\mathbf{b}^{e\,-1} + \mu\,J^{e\,-2/3}\,\Big(-\frac{1}{3}\,\mathbf{b}^e\times\mathbf{b}^{e\,-1} + \frac{1}{9}\,I_{\mathbf{b}^e}\,\mathbf{g}^{-1}\times\mathbf{b}^{e\,-1} + \frac{1}{2}\,(\mathbf{i}\otimes\mathbf{i} + \mathbf{i}\boxtimes\mathbf{i}) - \frac{1}{3}\,\mathbf{g}^{-1}\times\mathbf{g}\Big)$$

$$\frac{\partial \mathbf{g}\boldsymbol{\tau}}{\partial \mathbf{g}} \;= \frac{1}{2}\,\kappa\,(\mathbf{i}\times\mathbf{g}^{-1}) + \mu\,J^{e\,-2/3}\,\Big(\frac{1}{2}(\mathbf{i}\otimes\mathbf{b}^e + \mathbf{i}\boxtimes\mathbf{b}^e) - \frac{1}{3}\mathbf{g}\mathbf{b}^e\times\mathbf{g}^{-1} - \frac{1}{3}\mathbf{i}\times\mathbf{b}^e + \frac{1}{9}\,I_{\mathbf{b}^e}\,\mathbf{i}\times\mathbf{g}^{-1}\Big)$$

$$\frac{\partial \mathbf{g}\boldsymbol{\tau}}{\partial \mathbf{b}^e} \;= \frac{1}{2}\,\kappa\,\mathbf{i}\times\mathbf{b}^{e\,-1} + \mu\,J^{e\,-2/3}\,\Big(-\frac{1}{3}\,\mathbf{g}\mathbf{b}^e\times\mathbf{b}^{e\,-1} + \frac{1}{9}\,I_{\mathbf{b}^e}\,\mathbf{i}\times\mathbf{b}^{e\,-1} + \frac{1}{2}\,(\mathbf{g}\otimes\mathbf{i} + \mathbf{g}\boxtimes\mathbf{i}) - \frac{1}{3}\,\mathbf{i}\times\mathbf{g}\Big)$$

$$\mathbb{c} = \kappa\ ((\mathbf{g}^{-1} \times \mathbf{g}^{-1}) - lnJ^e\ (\mathbf{g}^{-1} \otimes \mathbf{g}^{-1} + \mathbf{g}^{-1} \boxtimes \mathbf{g}^{-1}))$$
$$+\frac{2}{3}\ \mu\ J^{e\,-2/3} \Big(-\mathbf{b}^e \times \mathbf{g}^{-1} + \frac{1}{3}\ I_{\mathbf{b}^e}\ \mathbf{g}^{-1} \times \mathbf{g}^{-1} - \mathbf{g}^{-1} \times \mathbf{b}^e$$
$$+\frac{1}{2}\ I_{\mathbf{b}^e}\ (\mathbf{g}^{-1} \otimes \mathbf{g}^{-1} + \mathbf{g}^{-1} \boxtimes \mathbf{g}^{-1}))$$

These relations expressed in component representation read as :

$$\tau^{ij} = \kappa\ \ln J^e\ g^{ij} + \mu\ J^{e\,-2/3} \left(\hat{G}^{ij} - \frac{1}{3}(g_{op}\hat{G}^{po})g^{ij}\right)$$

$$g\tau_{i\cdot}^{\ j} = \kappa\ lnJ^e\ \delta_i^j + \mu\ J^{e\,-2/3} \left(g_{im}\hat{G}^{mj} - \frac{1}{3}(g_{op}\hat{G}^{po})\delta_i^j\right)$$

$$\frac{\partial \boldsymbol{\tau}}{\partial \mathbf{b}^e}{}^{i\ \ \ l}_{\cdot jk\cdot} = \frac{1}{2}\ \kappa\ g^{il}\hat{G}_{jk} +\ \mu\ J^{e\,-2/3}\ (-\frac{1}{3}\ \hat{G}^{il}\hat{G}_{jk} + \frac{1}{9}\ (g_{op}\hat{G}^{po})g^{il}\hat{G}_{jk}$$
$$+\frac{1}{2}\ (\delta_j^i\delta_k^l + \delta_k^i\delta_j^l) -\ \frac{1}{3}\ g^{il}g_{jk})$$

$$\frac{\partial \mathbf{g}\boldsymbol{\tau}}{\partial \mathbf{g}}{}_{i\cdots}^{\ jkl} = \frac{1}{2}\ \kappa\ \delta_i^l g^{jk} + \mu\ J^{e\,-2/3}\ \Big(\frac{1}{2}\ (\delta_i^j\hat{G}^{kl} + \delta_i^k\hat{G}^{jl})$$
$$-\frac{1}{3}\ g_{im}\hat{G}^{ml}g^{jk} - \frac{1}{3}\ \delta_i^l\hat{G}^{jk} + \frac{1}{9}\ (g_{op}\hat{G}^{po})\delta_i^l g^{jk}\Big)$$

$$\frac{\partial \mathbf{g}\boldsymbol{\tau}}{\partial \mathbf{b}^e}{}_{ijk\cdot}^{\ \ \ \ l} = \frac{1}{2}\ \kappa\ \delta_i^l\hat{G}_{jk} +\ \mu\ J^{e\,-2/3}\ (-\frac{1}{3}\ g_{io}\hat{G}^{ol}\hat{G}_{jk} + \frac{1}{9}\ (g_{op}\hat{G}^{po})\delta_i^l\hat{G}_{jk}$$
$$+\frac{1}{2}\ (g_{ij}\delta_k^l + g_{ik}\delta_j^l) -\ \frac{1}{3}\ \delta_i^l g_{jk})$$

$$\mathbb{c}^{ijkl} = \kappa\ \big(g^{il}g^{jk} - lnJ^e(g^{ij}g^{kl} + g^{ik}g^{jl})\big) + \frac{2}{3}\ \mu\ J^{e\,-2/3} \Big(\frac{1}{2}\ (g_{op}\hat{G}^{po})(g^{ij}g^{kl} + g^{ik}g^{jl})$$
$$+\frac{1}{3}(g_{op}\hat{G}^{po})g^{il}g^{jk} - \hat{G}^{il}g^{jk} - g^{il}\hat{G}^{jk}\Big)$$

Note that we use here and in what follows a new differentiation convention in the following form:

$$\frac{\partial \mathbf{A}}{\partial \mathbf{B}} = \frac{\partial A_{ij}}{\partial B_{kl}}\ \mathbf{G}^i\ \otimes\ \mathbf{G}_k\ \otimes\ \mathbf{G}_l\ \otimes\ \mathbf{G}^j\ .$$

Also we use two new types of double contraction laws denoted by $\bullet\circ$ and $\circ\bullet$ where a filled circle $\bullet$ (unfilled circle $\circ$) represents contraction with the inner (outer) base vectors of a fourth-order tensor. Note that for a second-order tensor a distinction between inner and outer bases is irrelevant.

5.4.2 Alternative derivation of the evolution equations

The strain energy function is given by :

$$\psi = \psi^e(\mathbf{b}^e, \mathbf{g}) + \psi^p(\alpha) \tag{5.48}$$

Thus we obtain the dissipation inequality in the following form :

$$\begin{aligned}
&\boldsymbol{\tau} : \mathbf{d} - \dot{\psi} \geq 0 \\
&\Leftrightarrow \boldsymbol{\tau} : \mathbf{d} - \frac{\partial \psi}{\partial \mathbf{b}^e} : \dot{\mathbf{b}^e} - \frac{\partial \psi}{\partial \alpha}\dot{\alpha} \geq 0 \\
&\Leftrightarrow \boldsymbol{\tau} : \mathbf{d} - \frac{\partial \psi}{\partial \mathbf{b}^e} : (\mathrm{L}_{\mathbf{v}}(\mathbf{b}^e) + \mathbf{l}\mathbf{b}^e + \mathbf{b}_e\mathbf{l}^*) - \frac{\partial \psi}{\partial \alpha}\dot{\alpha} \geq 0 \\
&\Leftrightarrow \boldsymbol{\tau} : \mathbf{d} - \frac{\partial \psi}{\partial \mathbf{b}^e} : \mathrm{L}_{\mathbf{v}}(\mathbf{b}^e) - \mathbf{g}^{-1}\frac{\partial \psi}{\partial \mathbf{b}^e}\mathbf{b}^e : \mathbf{g}\mathbf{l} - \mathbf{b}^e\frac{\partial \psi}{\partial \mathbf{b}^e}\mathbf{g}^{-1} : \mathbf{l}^*\mathbf{g} - \frac{\partial \psi}{\partial \alpha}\dot{\alpha} \geq 0\,.
\end{aligned} \tag{5.49}$$

where, in turn, the material time derivative of the spatial metric $\mathbf{g}$ vanishes. Also, the Lie-derivative [Başar & Weichert 2000] has been used. If we now consider an isotropic hyperelastic law the following holds

$$\mathbf{g}^{-1}\frac{\partial \psi}{\partial \mathbf{b}^e}\mathbf{b}^e = \mathbf{b}^e\frac{\partial \psi}{\partial \mathbf{b}^e}\mathbf{g}^{-1} = \frac{\partial \psi}{\partial \mathbf{g}}\,, \tag{5.50}$$

such that with the definition for the strain rate $\mathbf{d} = 1/2(\mathbf{g}\mathbf{l} + \mathbf{l}^*\mathbf{g})$ we obtain :

$$(\boldsymbol{\tau} - 2\,\mathbf{g}^{-1}\frac{\partial \psi}{\partial \mathbf{b}^e}\mathbf{b}^e) : \mathbf{d} - \frac{\partial \psi}{\partial \mathbf{b}^e} : \mathrm{L}_{\mathbf{v}}(\mathbf{b}^e) - \frac{\partial \psi}{\partial \alpha}\dot{\alpha} \geq 0\,. \tag{5.51}$$

At first, we get the hyperelastic material law :

$$\boldsymbol{\tau} = 2\,\mathbf{g}^{-1}\frac{\partial \psi}{\partial \mathbf{b}^e}\mathbf{b}_e = 2\,\frac{\partial \psi}{\partial \mathbf{g}}\,, \tag{5.52}$$

and a reduced dissipation inequality

$$-\frac{\partial \psi}{\partial \mathbf{b}^e} : \mathrm{L}_{\mathbf{v}}(\mathbf{b}^e) - \frac{\partial \psi}{\partial \alpha}\dot{\alpha} \geq 0\,, \tag{5.53}$$

from which by considering **Equation** (5.52) we get :

$$-\frac{1}{2}\,\mathbf{g}\boldsymbol{\tau} : \mathrm{L}_{\mathbf{v}}(\mathbf{b}^e)\mathbf{b}^{e\,-1} - q\dot{\alpha} \geq 0\,, \tag{5.54}$$

with

$$q = \frac{\partial \psi}{\partial \alpha}\,. \tag{5.55}$$

The yield criterion is assumed as

$$F = \sqrt{dev(\mathbf{g}\boldsymbol{\tau})dev(\mathbf{g}\boldsymbol{\tau}) : \mathbf{i}} - \sqrt{\frac{2}{3}}q(\alpha) \leq 0\,. \tag{5.56}$$

Now, by means of normality rules emanating from the principle of maximum dissipation, we get the following evolution equations :

$$\begin{aligned} \frac{1}{2}\mathrm{L}_{\mathbf{v}}(\mathbf{b}^e)\mathbf{b}^{e\,-1} &= -\dot{\gamma}\mathbf{n} = -\dot{\gamma}\frac{\partial F}{\partial(\mathbf{g}\boldsymbol{\tau})}\,, \\ \dot{\alpha} &= \dot{\gamma}\sqrt{\frac{2}{3}}\,. \end{aligned} \tag{5.57}$$

The above evolution equations can be integrated according to the following rules:

$$\mathbf{b}^e_{(n+1)} = \exp(-\Delta\gamma\, \mathbf{n}_{(n+1)})\mathbf{b}^e_{(trial)} \exp(-\Delta\gamma \mathbf{n}^*_{(n+1)})\,, \tag{5.58}$$

and

$$\alpha_{(n+1)} = \alpha_{(n)} + \Delta\gamma\sqrt{\frac{2}{3}}\,, \tag{5.59}$$

where $\mathbf{b}^e_{(trial)}$ is simply the push-forward of $\mathbf{b}^e_{(n)}$:

$$\mathbf{b}^e_{(trial)} = \mathbf{F}_{\triangleright\,(n+1)}\left(\mathbf{F}^{\triangleleft}_{(n)}\,(\mathbf{b}^e_{(n)})\right)\,, \tag{5.60}$$

with the component expressions

$$\mathbf{b}^e_{(trial)} \;=\; \hat{G}^{ij}_{(n)}\mathbf{g}_i \otimes \mathbf{g}_j, \qquad \mathbf{b}^e_{(n+1)} \;=\; \hat{G}^{ij}_{(n+1)}\mathbf{g}_i \otimes \mathbf{g}_j \tag{5.61}$$

The discriminand of the yield function reads as:

$$\mathrm{tr}((\mathbf{g}\boldsymbol{\tau})^2) - \frac{1}{3}(\mathrm{tr}(\mathbf{g}\boldsymbol{\tau}))^2\,. \tag{5.62}$$

From that we obtain the normal to the yield surface in the form :

$$\mathbf{n} = \frac{\partial F}{\partial(\mathbf{g}\boldsymbol{\tau})} = \frac{1}{\sqrt{\cdots}}\left((\mathbf{g}\boldsymbol{\tau})^T - \frac{1}{3}\,tr(\mathbf{g}\boldsymbol{\tau})\mathbf{i}\right)\,. \tag{5.63}$$

In this case the exponential law **Equation** (5.58) accomplishes an update of the metric of the intermediate configuration from state (n) to the current state $(n+1)$.

5.4.3 Isotropic plastic model in eigenvalues

Now we want to present a model for isotropic plasticity, which is based on the previous model but is formulated in eigenvalues. The important point is that once the trial-eigenprojections are obtained they are constant during the entire iteration such that only a reduced model in terms of the eigenvalues results. As elastic model we use a model previously introduced (see **Equation** (4.14)) with a volumetric-isochoric split. At first we want to give the relevant equations in absolute tensor notation :
Elastic potential :

$$\psi^e = \frac{1}{2}\,\kappa\,(lnJ^e)^2 + \frac{1}{2}\,\mu\,(J^{e\,-2/3}\;tr(\mathbf{g}\mathbf{b}^e) - 3) \tag{5.64}$$

with $J^e = \sqrt{det(\mathbf{b}^e\mathbf{g})}$.
KIRCHHOFF-stress :

$$\boldsymbol{\tau} = 2\,\frac{\partial\psi^e}{\partial\mathbf{g}} = \kappa\,\ln J^e\;\mathbf{g}^{-1} + \mu\;J^{e\,-2/3}\left(\mathbf{b}^e - \frac{1}{3}\,\mathrm{tr}(\mathbf{g}\mathbf{b}^e)\;\mathbf{g}^{-1}\right) \tag{5.65}$$

Stress deviator :

$$\mathrm{dev}(\boldsymbol{\tau}) = \mu\;J^{e\,-2/3}\left(\mathbf{b}^e - \frac{1}{3}\,\mathrm{tr}(\mathbf{g}\mathbf{b}^e)\;\mathbf{g}^{-1}\right) \tag{5.66}$$

Yield function :

$$\begin{aligned} F &= \sqrt{\mathrm{tr}((\mathrm{dev}(\mathbf{g}\boldsymbol{\tau}))^2)} - \sqrt{\frac{2}{3}}\,q(\alpha) \\ &= \mu\;J^{e\,-2/3}\sqrt{\mathrm{tr}(\mathbf{g}\mathbf{b}^e\mathbf{g}\mathbf{b}^e) - \frac{1}{3}\,(\mathrm{tr}(\mathbf{g}\mathbf{b}^e))^2} - \sqrt{\frac{2}{3}}\,q(\alpha) \leq 0 \end{aligned} \tag{5.67}$$

Normal to the yield surface :

$$\mathbf{n} = \frac{\partial F}{\partial(\mathbf{g}\boldsymbol{\tau})} = \frac{1}{\sqrt{\mathrm{tr}((\mathrm{dev}(\mathbf{g}\boldsymbol{\tau}))^2)}}\;\mathrm{dev}(\boldsymbol{\tau}\mathbf{g}) = \frac{\mathbf{b}^e\mathbf{g} - \frac{1}{3}\,\mathrm{tr}(\mathbf{g}\mathbf{b}^e)\;\mathbf{i}}{\sqrt{\mathrm{tr}(\mathbf{g}\mathbf{b}^e\mathbf{g}\mathbf{b}^e) - \frac{1}{3}\;(\mathrm{tr}(\mathbf{g}\mathbf{b}^e))^2}} \tag{5.68}$$

Elastic stiffness matrix :

$$\begin{aligned} \mathbb{c} = \;& \kappa\,((\mathbf{g}^{-1}\times\mathbf{g}^{-1}) - \ln J^e\;(\mathbf{g}^{-1}\otimes\mathbf{g}^{-1} + \mathbf{g}^{-1}\boxtimes\mathbf{g}^{-1})) \\ & + \frac{2}{3}\,\mu\;J^{e\,-2/3}\left(-\mathbf{b}^e\times\mathbf{g}^{-1} - \mathbf{g}^{-1}\times\mathbf{b}^e + \frac{1}{3}\,I_{\mathbf{b}^e}\;\mathbf{g}^{-1}\times\mathbf{g}^{-1}\right. \\ & \left. + \frac{1}{2}\,I_{\mathbf{b}^e}\;(\mathbf{g}^{-1}\otimes\mathbf{g}^{-1} + \mathbf{g}^{-1}\boxtimes\mathbf{g}^{-1})\right) \end{aligned} \tag{5.69}$$

Evolution equation for $\mathbf{b}_e$:

$$\mathbf{b}^e_{(n+1)} = exp(-\Delta\gamma\mathbf{n})\mathbf{b}^e_{(trial)} exp(-\Delta\gamma\mathbf{n}^*) \tag{5.70}$$

Now we want to give the equations exploiting spectral decompositions of all tensors. As a well-known fact the elastic left CAUCHY-GREEN-tensor and the metric tensor related to the current configuration can be decomposed in the form :

$$\mathbf{b}^e = \sum_{i=1}^{3} \lambda_i^2 \, \mathbf{n}_i \otimes \mathbf{n}_i, \qquad \mathbf{g}^{-1} = \mathbf{g} = \mathbf{i} = \sum_{i=1}^{3} \mathbf{n}_i \otimes \mathbf{n}_i \tag{5.71}$$

From the evolution law for $\mathbf{b}_e$ from **Equation** (5.70) one obviously sees that under the postulate of the uniqueness of the spectral decomposition, the eigenprojections of $\mathbf{b}^e$ are invariant during the return-map:

$$\mathbf{n}_{i\,(trial)} \otimes \mathbf{n}_{i\,(trial)} = \mathbf{n}_{i\,(n+1)} \otimes \mathbf{n}_{i\,(n+1)} = \mathbf{n}_{i\,(i)} \otimes \mathbf{n}_{i\,(i)} \text{ (no summation over i)} \tag{5.72}$$

with $\mathbf{n}_{i\,(trial)} \otimes \mathbf{n}_{i\,(trial)}$ as the trial-eigenprojection and $\mathbf{n}_{i\,(i)} \otimes \mathbf{n}_{i\,(i)}$ as the eigenprojection during the return-map in iteration step i. Therefore only the eigenvalues of $\mathbf{b}^e$ are variable. The tensors are given by :
KIRCHHOFF-stress tensor :

$$\boldsymbol{\tau} = \sum_{i=1}^{3} \left(\kappa \, lnJ^e + \mu \, J^{e\,-2/3} \left(\lambda_i^2 - \frac{1}{3} \left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 \right) \right) \right) \mathbf{n}_i \otimes \mathbf{n}_i \tag{5.73}$$

with $J^e = \sqrt{\lambda_1^2 \lambda_2^2 \lambda_3^2}$.
The stress deviator :

$$dev(\boldsymbol{\tau}) = \sum_{i=1}^{3} \mu \, J^{e\,-2/3} \left(\lambda_i^2 - \frac{1}{3}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) \right) \mathbf{n}_i \otimes \mathbf{n}_i \tag{5.74}$$

The yield function :

$$\begin{aligned} F &= \mu \, J^{e\,-2/3} \sqrt{\lambda_1^4 + \lambda_2^4 + \lambda_3^4 - \tfrac{1}{3} \left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 \right)^2} - \sqrt{\tfrac{2}{3}} \, q(\alpha) \\ &= \mu \, J^{e\,-2/3} \sqrt{\tfrac{2}{3} \left(\lambda_1^4 + \lambda_2^4 + \lambda_3^4 - \lambda_1^2\lambda_2^2 - \lambda_1^2\lambda_3^2 - \lambda_2^2\lambda_3^2 \right)} - \sqrt{\tfrac{2}{3}} \, q(\alpha) \end{aligned} \tag{5.75}$$

The normal to the yield surface :

$$\mathbf{n} = \sum_{i=1}^{3} n_i \, \mathbf{n}_i \otimes \mathbf{n}_i = \sum_{i=1}^{3} \frac{\lambda_i^2 - \frac{1}{3} \left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 \right)}{\sqrt{\frac{2}{3} \left(\lambda_1^4 + \lambda_2^4 + \lambda_3^4 - \lambda_1^2\lambda_2^2 - \lambda_1^2\lambda_3^2 - \lambda_2^2\lambda_3^2 \right)}} \, \mathbf{n}_i \otimes \mathbf{n}_i \tag{5.76}$$

The evolution law for the elastic left CAUCHY-GREEN-tensor:

$$\begin{aligned}
\mathbf{b}^e_{(n+1)} &= exp(-\Delta\gamma_{(n+1)}\mathbf{n}_{(n+1)})\mathbf{b}^e_{(trial)}exp(-\Delta\gamma_{(n+1)}\mathbf{n}^*_{(n+1)}) \\
\sum_{i=1}^{3}\lambda^2_{i\,(n+1)}\mathbf{n}_i\otimes\mathbf{n}_i &= \textstyle\sum_{i=1}^{3} exp(-2\Delta\gamma_{(n+1)}n_{i\,(n+1)})\lambda^2_{i\,(trial)}\;\mathbf{n}_i\otimes\mathbf{n}_i
\end{aligned} \tag{5.77}$$

Chapter 6

Algorithmic setting

6.1 Introduction

The above problem is highly nonlinear by nature. Therefore to solve this problem we have to employ a NEWTON-RAPHSON-procedure on Gauss-point level. There are two unknowns in the coupled system: $\mathbf{n}_{(n+1)}$ and $\Delta\gamma$. To get a solution for these values we have to consider certain auxiliary conditions from which these values can be derived. Since the yield function F is a scalar equation it is used as auxiliary condition to derive the consistency parameter. Normally $F > 0$ is not allowed. However, in the actual computations it is possible that we have elastic behaviour and end up somewhere beyond the elastic domain where $F > 0$. Since $F \leq 0$ has to be fullfilled anyway we have to project the stress tensor from beyond the yield condition exactly onto the yield condition, so that $F = 0$ holds. As we can see from **Equation** (5.45) the direction of this projection is the direction of the normal $\mathbf{n}_{(n+1)}$ to the yield surface. The consistency parameter measures the scalar length of the projection $\mathbf{n}_{(n+1)}$ onto the yield surface. Therefore by using F as auxiliary condition and by demanding $F = 0$ we can compute the consistency parameter $\Delta\gamma$. But we have to find also an auxiliary condition for the normal itself. To compute the normal we just use the definition for the normal in residual form :

$$\mathbf{R_n} = \mathbf{n}_{(i)} - \frac{\partial F_{(i)}}{\partial(\mathbf{g}\boldsymbol{\tau})_{(i)}} = 0 \tag{6.1}$$

which has to be fulfilled iteratively. Note that due to the nonlinear nature of the problem $\mathbf{n}_{(n+1)}$ is not equal to $\mathbf{n}_{(trial)}$.

6.1.1 Return-map in tensor notation

The physical nonlinear equation system will be solved using a NEWTON-method. As residua we use on the one hand the yield criterion and the normal to the yield surface $\mathbf{n}_{(n+1)}$:

1. The yield criterion :

$F_{(i)}(\mathbf{g}\boldsymbol{\tau}_{(i)}(\mathbf{g}, \mathbf{b}^e_{(i)}(\bar{\mathbf{n}}_{(i)}, \Delta\gamma^i)), q(\Delta\gamma^i)) = 0$

2. The conditional equation for the normal :

$\mathbf{R}_{\mathbf{n}\,(i)} = \bar{\mathbf{n}}_{(i)} - \frac{1}{\sqrt{\cdots}}\left((\mathbf{g}\boldsymbol{\tau})^T_{(i)} - \frac{1}{3}\,\mathrm{tr}(\mathbf{g}\boldsymbol{\tau})_{(i)}\,\mathbf{i}\right) = \mathbf{0}$

The zero values for both equations are obtained by a NEWTON-method :

$$\mathbf{h}_{(i)} + \mathbf{h}_{,\mathbf{y}} \cdot \Delta\mathbf{y} = \mathbf{0}\,, \tag{6.2}$$

with

$$\mathbf{h}_{,\mathbf{y}} = \begin{pmatrix} \dfrac{\partial F_{(i)}}{\partial \Delta\gamma} & \dfrac{\partial F_{(i)}}{\partial \bar{\mathbf{n}}_{(i)}} \\ \dfrac{\partial \mathbf{R}_{\mathbf{n}\,(i)}}{\partial \Delta\gamma} & \dfrac{\partial \mathbf{R}_{\mathbf{n}\,(i)}}{\partial \bar{\mathbf{n}}_{(i)}} \end{pmatrix} \tag{6.3}$$

Afterwards the following quantities are updated :

$$\Delta\mathbf{y} = -\mathbf{h}_{,\mathbf{y}}^{-1} \cdot \mathbf{h}_{(i)}\ , \qquad \mathbf{y}_{(i+1)} = \mathbf{y}_{(i)} + \Delta\mathbf{y}\,. \tag{6.4}$$

The solution presupposes the construction of the following vector in terms of the basic unknowns $\Delta\gamma$ and $\mathbf{n}_{(i)}$:

$$\begin{aligned}\mathbf{y} = \big(\Delta\gamma,\ &\bar{n}_{(i)}{}^1_{.\,1},\ \bar{n}_{(i)}{}^2_{.\,2},\ \bar{n}_{(i)}{}^3_{.\,3},\ \bar{n}_{(i)}{}^1_{.\,2},\ \bar{n}_{(i)}{}^1_{.\,3},\\ &\bar{n}_{(i)}{}^2_{.\,3},\ \bar{n}_{(i)}{}^2_{.\,1},\ \bar{n}_{(i)}{}^3_{.\,1},\ \bar{n}_{(i)}{}^3_{.\,2}\big)\ .\end{aligned} \tag{6.5}$$

$$\begin{aligned}\mathbf{h}_{(i)} = \big(F_{(i)},\ &\mathbf{R}_{\mathbf{n}\,(i)}{}^1_{.\,1},\ \mathbf{R}_{\mathbf{n}\,(i)}{}^2_{.\,2},\ \mathbf{R}_{\mathbf{n}\,(i)}{}^3_{.\,3},\ \mathbf{R}_{\mathbf{n}\,(i)}{}^1_{.\,2},\ \mathbf{R}_{\mathbf{n}\,(i)}{}^1_{.\,3},\\ &\mathbf{R}_{\mathbf{n}\,(i)}{}^2_{.\,3},\ \mathbf{R}_{\mathbf{n}\,(i)}{}^2_{.\,1},\ \mathbf{R}_{\mathbf{n}\,(i)}{}^3_{.\,1},\ \mathbf{R}_{\mathbf{n}\,(i)}{}^3_{.\,2}\big)\end{aligned} \tag{6.6}$$

where $\Delta\gamma = 0$ and $\bar{n}_{(0)} = n_{(trial)}$ at the beginning. Since the component matrix of $\mathbf{n}_{(n+1)}$ is in general unsymmetric we have to use 9 components. The single equations to be used in **Equation** (6.3) read as :

$$\begin{aligned}
\frac{\partial F_{(i)}}{\partial \bar{\mathbf{n}}_{(i)}} &= \frac{\partial F_{(i)}}{\partial (\mathbf{g}\boldsymbol{\tau})_{(i)}} : \frac{\partial (\mathbf{g}\boldsymbol{\tau})_{(i)}}{\partial \mathbf{b}^e_{(i)}} : \frac{\partial \mathbf{b}^e_{(i)}}{\partial \bar{\mathbf{n}}_{(i)}} \\
\frac{\partial F_{(i)}}{\partial \Delta\gamma} &= \frac{\partial F_{(i)}}{\partial (\mathbf{g}\boldsymbol{\tau})_{(i)}} : \frac{\partial (\mathbf{g}\boldsymbol{\tau})_{(i)}}{\partial \mathbf{b}^e_{(i)}} : \frac{\partial \mathbf{b}^e_{(i)}}{\partial \Delta\gamma} + \frac{\partial F_{(i)}}{\partial q_{(i)}} \frac{\partial q_{(i)}}{\partial \alpha_{(i)}} \frac{\partial \alpha_{(i)}}{\partial \Delta\gamma} \\
\frac{\partial \mathbf{R}_{\mathbf{n}\,(i)}}{\partial \bar{\mathbf{n}}_{(i)}} &= \mathbf{i} \otimes \mathbf{i} + \frac{\partial \mathbf{R}_{\mathbf{n}\,(i)}}{\partial (\mathbf{g}\boldsymbol{\tau})_{(i)}} : \frac{\partial (\mathbf{g}\boldsymbol{\tau})_{(i)}}{\partial \mathbf{b}^e_{(i)}} : \frac{\partial \mathbf{b}^e_{(i)}}{\partial \bar{\mathbf{n}}_{(i)}} \\
\frac{\partial \mathbf{R}_{\mathbf{n}\,(i)}}{\partial \Delta\gamma} &= \frac{\partial \mathbf{R}_{\mathbf{n}\,(i)}}{\partial (\mathbf{g}\boldsymbol{\tau})_{(i)}} : \frac{\partial (\mathbf{g}\boldsymbol{\tau})_{(i)}}{\partial \mathbf{b}^e_{(i)}} : \frac{\partial \mathbf{b}^e_{(i)}}{\partial \Delta\gamma} \,.
\end{aligned} \tag{6.7}$$

One advantage of such a description is, that the basis $\mathbf{g}_i$ stays fixed during the return-map. Therefore no component transformations are necessary during the return-map:

$$\begin{aligned}
\frac{\partial F_{(i)}}{\partial (\mathbf{g}\boldsymbol{\tau})_{(i)}} &= \mathbf{n}_{(n+1)} \\
\frac{\partial \mathbf{R}_{\mathbf{n}\,(i)}}{\partial (\mathbf{g}\boldsymbol{\tau})_{(i)}} &= -\tfrac{1}{\sqrt{\cdots}} \left(\mathbf{i} \boxtimes \mathbf{i} - \tfrac{1}{3}\,\mathbf{i} \times \mathbf{i} - \mathbf{n}_{(i)} \times \mathbf{n}_{(i)}\right) \\
\frac{\partial \mathbf{b}^e_{(i)}}{\partial \bar{\mathbf{n}}_{(i)}} &= -\Delta\gamma \Big[\exp(-\Delta\gamma \bar{\mathbf{n}}_{(i)})_{,(-\Delta\gamma\bar{\mathbf{n}}_{(i)})}\, \mathbf{b}^e_{(trial)} \exp(-\Delta\gamma \bar{\mathbf{n}}^*_{(i)}) \\
&\qquad + \exp(-\Delta\gamma \bar{\mathbf{n}}_{(i)}) \mathbf{b}^e_{(trial)} \exp(-\Delta\gamma \bar{\mathbf{n}}^*_{(i)})_{,(-\Delta\gamma\bar{\mathbf{n}}_{(i)})}\Big] \\
\frac{\partial \mathbf{b}^e_{(i)}}{\partial \Delta\gamma} &= -\exp(-\Delta\gamma \bar{\mathbf{n}}_{(i)}) \Big[\bar{\mathbf{n}}_{(i)} \mathbf{b}^e_{(trial)} + \mathbf{b}^e_{(trial)} \bar{\mathbf{n}}^*_{(i)}\Big] \exp(-\Delta\gamma \bar{\mathbf{n}}^*_{(i)}) \\
\frac{\partial F_{(i)}}{\partial q_{(i)}} \frac{\partial q_{(i)}}{\partial \alpha_{(i)}} \frac{\partial \alpha_{(i)}}{\partial \Delta\gamma} &= -\delta\, \frac{2}{3}\, (\sigma_{Y\infty} - \sigma_{Y0})\, e^{(-\delta\,\alpha_{(i)})} - \frac{2}{3}\, H \qquad \text{law 1:} \\
\frac{\partial F_{(i)}}{\partial q_{(i)}} \frac{\partial q_{(i)}}{\partial \alpha_{(i)}} \frac{\partial \alpha_{(i)}}{\partial \Delta\gamma} &= -\frac{2}{3}\, a_1\, a_3\, \left(a_2 + \alpha_{(i)}\right)^{a_3 - 1} \qquad \text{law 2:}
\end{aligned} \tag{6.8}$$

if we consider in extension of rule **Equation** (5.35) the following isotropic hardening laws:

$$\begin{aligned}
q(\alpha) &= \sigma_{Y_0} + (\sigma_\infty - \sigma_{Y_0})(1 - \exp(-\delta\,\alpha)) + H\,\alpha \qquad \text{law 1:} \\
q(\alpha) &= a_1\,(a_2 + \alpha)^{a_3} \qquad \text{law 2:}
\end{aligned} \tag{6.9}$$

The iteration is done until the residua $F_{(i)} = 0$ and $\mathbf{R}_{\mathbf{n}\,(i)} = 0$ are fulfilled up to a certain tolerance limit which is usually set to 10^{-12}. The end values of the NEWTON-procedure are set to $\Delta\gamma$ and $\mathbf{n}_{(n+1)}$ with which the stresses as output values can be computed. As further output values for the global solution we also need the stiffness matrix that means the consistent elasto-plastic tangent operator $\mathbb{C}^{ep}$. This will be discussed in a forthcoming section.

6.2 Computation of eigenvalues and eigenprojections

As starting point of the algorithm in principal axes we have to compute the eigenvalues and eigenprojections of $\mathbf{b}^e$.

Note that only two tensors are used in the elastic law: $\mathbf{b}^e$ and $\mathbf{g}$. Both tensors are symmetric and therefore have a certain decomposition with real principal values. Since $\mathbf{g}$ is an identity tensor in the related spectral decomposition its eigenvalues are 1.

$$
\begin{aligned}
\mathbf{b}^e &= \hat{G}^{ij}\mathbf{g}_i \otimes \mathbf{g}_j = \lambda_i^2 \mathbf{n}_i \otimes \mathbf{n}_i = \lambda_i^2 \mathbf{m}_i \,, \\
\mathbf{g} &= g_{ij}\mathbf{g}_i \otimes \mathbf{g}_j = \delta_{ij}\,\mathbf{n}_i \otimes \mathbf{n}_j = \sum_{i=1}^{3} \mathbf{m}_i \,,
\end{aligned}
\tag{6.10}
$$

where $\mathbf{m}_i$ are the eigenprojections and $\mathbf{g}_i \otimes \mathbf{g}_j$ is the tensor basis. In subsequent computations we use the eigenprojections instead of the tensor basis. For the problem we have to compute the eigenvalues of the tensor $\mathbf{b}^e$. At first note that the components of $\mathbf{b}^e$ have been resolved with respect to the basis $\mathbf{g}_i$ which is deformation dependent. To compute the eigenvalues of $\mathbf{b}^e$ we have at first to resolve $\mathbf{b}^e$ with respect to a basis $\mathbf{i}_i$ fixed in space. Using the shell kinematics the basis $\mathbf{g}_i$ is defined by :

$$
\begin{aligned}
\mathbf{g}_i &= \overset{0}{\mathbf{x}}_{,\theta^i} + \theta^3(\lambda_{,\theta^i}\mathbf{d} + \lambda\mathbf{d}_{,\theta^i}) \,, \qquad \text{for i} = 1,2 \,, \\
\mathbf{g}_3 &= \lambda\mathbf{d} \,,
\end{aligned}
\tag{6.11}
$$

and similarly for the reference configuration:

$$
\begin{aligned}
\mathbf{G}_i &= \overset{0}{\mathbf{X}}_{,\theta^i} + \theta^3(\mathbf{D}_{,\theta^i}), \qquad \text{for i} = 1,2 \,, \\
\mathbf{G}_3 &= \mathbf{D} \,.
\end{aligned}
\tag{6.12}
$$

Then the components of $\mathbf{b}^e$ with respect to the basis $\mathbf{i}_i$ can be computed by

$$
\mathbf{b}^e_{ij} = (\mathbf{i}_i \cdot \mathbf{b}^e) \cdot \mathbf{i}_j = (\mathbf{i}_i \cdot (\hat{\mathbf{G}}^{kl}\mathbf{g}_k \otimes \mathbf{g}_l)) \cdot \mathbf{i}_j = \hat{\mathbf{G}}^{kl}(\mathbf{g}_k \cdot \mathbf{i}_i)(\mathbf{g}_l \cdot \mathbf{i}_j) \,. \tag{6.13}
$$

As previously mentioned we use an enhanced assumed strain formulation as element stabilization. The incompatible modes are added to the strains that means the tensor components $\bar{g}_{ij}$ contain both compatible and incompatible modes. However, the basis $\mathbf{g}_i$ is solely computed from the kinematical values, and the shell kinematics does depend only on compatible deformations. That means the basis $\mathbf{g}_i$ obtainable from the displacement field is not the real basis we wish to have. Therefore we have to compute the real basis which we call $\bar{\mathbf{g}}_i$. To compute this basis we start from the definition for the deformation gradient $\mathbf{F} = \mathbf{RU}$ which is explained in the **Section 4.4.2**. At first we compute the components of the deformation gradient in the reference basis (see **Section 2.6.2**):

$$\mathbf{F} = \mathbf{g}_i \otimes \mathbf{G}^i = F^i_{.j}\,\mathbf{G}_i \otimes \mathbf{G}^i \Rightarrow F^i_{.j} = (\mathbf{G}^i \cdot (\mathbf{g}_k \otimes \mathbf{G}^k)) \cdot \mathbf{G}_j = \delta^k_{.j}(\mathbf{G}^i \cdot \mathbf{g}_k) = \mathbf{G}^i \cdot \mathbf{g}_j \,. \tag{6.14}$$

Next we compute the metric tensor components g_{ij} solely from the displacement field by disregarding the incompatible modes. From this we compute the right stretch tensor $\mathbf{U}$ which delivers the rotation tensor $\mathbf{R} = \mathbf{F}\mathbf{U}^{-1}$ (Note that we use the decompositions $\mathbf{R} = R^i_{.j}\,\mathbf{G}_i \otimes \mathbf{G}^j$ and $\mathbf{U} = U^i_{.j}\,\mathbf{G}_i \otimes \mathbf{G}^j$). Afterwards we do the same for the real metric tensor $\bar{g}_{ij}$ which includes both compatible and incompatible stretches, from which we derive the stretch tensor $\bar{\mathbf{U}}$. Then we compute the real deformation gradient:

$$\bar{\mathbf{F}} = \mathbf{R}\bar{\mathbf{U}} = \bar{F}^i_{.j}\,\mathbf{G}_i \otimes \mathbf{G}^j \tag{6.15}$$

Then we obtain the real basis $\bar{\mathbf{g}}_i$ using the above deformation gradient:

$$\bar{\mathbf{g}}_i = \bar{\mathbf{F}}\,\mathbf{G}_i = (\bar{F}^k_{.l}\,\mathbf{G}_k \otimes \mathbf{G}^l) \cdot \mathbf{G}_i = \bar{F}^k_{.l}\,(\mathbf{G}^l \cdot \mathbf{G}_i)\,\mathbf{G}_k = \bar{F}^k_{.i}\,\mathbf{G}_k \,. \tag{6.16}$$

Thus the components of $\mathbf{b}^e$ with respect to the "real" basis are given by

$$\mathbf{b}^e = \hat{\mathbf{G}}^{ij}\bar{\mathbf{g}}_i \otimes \bar{\mathbf{g}}_j \Rightarrow \mathbf{b}^e_{ij} = \hat{\mathbf{G}}^{kl}\,(\bar{\mathbf{g}}_k \cdot \mathbf{i}_i)(\bar{\mathbf{g}}_l \cdot \mathbf{i}_j) \tag{6.17}$$

6.3 Return-map in principal axes

In case of an implicit rule the normal and the plastic consistency parameter are implicitly determined. The following dependencies exist :

$$\begin{aligned} n_{i\,(i)} &= n_{i\,(i)}\left(\lambda^2_{i\,(i)}\left(\Delta\gamma_{(i)}, \lambda^2_{i\,(trial)}, \bar{n}_{i\,(i)}\right)\right) \\ F_{(i)} &= F_{(i)}\left(\lambda^2_{i\,(i)}\left(\Delta\gamma_{(i)}, \lambda^2_{i\,(trial)}, \bar{n}_{i\,(i)}\right), q_{(i)}(\alpha_{(i)}(\Delta\gamma_{(i)}))\right) \end{aligned} \tag{6.18}$$

Therefore we obtain 4 equations to determine the unknowns $n_{i\,(n+1)}$ and $\Delta\gamma_{(n+1)}$. To compute these values we employ a Newton-procedure :

$$\mathbf{h}_{(i)} + \mathbf{h}_{,\mathbf{y}} \cdot \Delta\mathbf{y} = \mathbf{0} \tag{6.19}$$

with

$$\mathbf{h}_{,\mathbf{y}} = \begin{pmatrix} \dfrac{\partial F_{(i)}}{\partial \Delta\gamma_{(i)}} & \dfrac{\partial F_{(i)}}{\partial \bar{n}_{i\,(i)}} \\ \dfrac{\partial \mathbf{R}_{k\,(i)}}{\partial \Delta\gamma_{(i)}} & \dfrac{\partial \mathbf{R}_{k\,(i)}}{\partial \bar{n}_{i\,(i)}} \end{pmatrix} \tag{6.20}$$

the residuals

$$\mathbf{R}_{k\,(i)} = \bar{n}_{k\,(i)} - \frac{\lambda^2_{k\,(i)} - \frac{1}{3}(\lambda^2_{1\,(i)} + \lambda^2_{2\,(i)} + \lambda^2_{3\,(i)})}{\sqrt{\frac{2}{3}(\lambda^4_{1\,(i)} + \lambda^4_{2\,(i)} + \lambda^4_{3\,(i)} - \lambda^2_{1\,(i)}\lambda^2_{2\,(i)} - \lambda^2_{1\,(i)}\lambda^2_{3\,(i)} - \lambda^2_{2\,(i)}\lambda^2_{3\,(i)})}} = 0 \tag{6.21}$$

and the start vector

$$\mathbf{y}_{(0)} = \{0, \bar{n}_{1\,(trial)}, \bar{n}_{2\,(trial)}, \bar{n}_{3\,(trial)}\}\,, \qquad \mathbf{h}_{(i)} = \{F_{(i)}, \mathbf{R}_{1\,(i)}, \mathbf{R}_{2\,(i)}, \mathbf{R}_{3\,(i)}\} \tag{6.22}$$

Note that $\bar{n}_{k\,(i)}$ denotes the iterated value of the normal while $n_{k\,(i)}$ are the exact components obtained by means of the implicit law **Equation** (5.76). The linearization finally delivers :

$$\Delta\mathbf{y} = -\mathbf{h}^{-1}_{,\mathbf{y}} \cdot \mathbf{h}_{(i)}\ , \qquad \mathbf{y}_{(i+1)} = \mathbf{y}_{(i)} + \Delta\mathbf{y} \tag{6.23}$$

The tensors appearing above are determined by :

$$\begin{aligned}
\frac{\partial F_{(i)}}{\partial \Delta\gamma_{(i)}} &= \frac{\partial F_{(i)}}{\partial \lambda^2_{i\,(i)}}\frac{\partial \lambda^2_{i\,(i)}}{\partial \Delta\gamma_{(i)}} + \frac{\partial F_{(i)}}{\partial q_{(i)}}\frac{\partial q_{(i)}}{\partial \alpha_{(i)}}\frac{\partial \alpha_{(i)}}{\partial \Delta\gamma_{(i)}} \\
\frac{\partial F_{(i)}}{\partial \bar{n}_{i\,(i)}} &= \frac{\partial F_{(i)}}{\partial \lambda^2_{m\,(i)}}\frac{\partial \lambda^2_{m\,(i)}}{\partial \bar{n}_{i\,(i)}} \\
\frac{\partial \mathbf{R}_{k\,(i)}}{\partial \Delta\gamma_{(i)}} &= -\frac{\partial n_{k\,(i)}}{\partial \lambda^2_{m\,(i)}}\frac{\partial \lambda^2_{m\,(i)}}{\partial \Delta\gamma_{(i)}} \\
\frac{\partial \mathbf{R}_{k\,(i)}}{\partial \bar{n}_{i\,(i)}} &= \delta_{ki} - \frac{\partial n_{k\,(i)}}{\partial \lambda^2_{m\,(i)}}\frac{\partial \lambda^2_{m\,(i)}}{\partial \bar{n}_{i\,(i)}}
\end{aligned} \tag{6.24}$$

with

$$\begin{aligned}
\frac{\partial F_{(i)}}{\partial \lambda^2_{i\,(i)}} = \mu\, J^{e\,-2/3} \Bigg(& \frac{\left(\lambda^2_{i\,(i)} - \frac{1}{3}(\lambda^2_{1\,(i)} + \lambda^2_{2\,(i)} + \lambda^2_{3\,(i)})(\delta_{1i} + \delta_{2i} + \delta_{3i})\right)}{\sqrt{\frac{2}{3}\left(\lambda^4_{1\,(i)} + \lambda^4_{2\,(i)} + \lambda^4_{3\,(i)} - \lambda^2_{1\,(i)}\lambda^2_{2\,(i)} - \lambda^2_{1\,(i)}\lambda^2_{3\,(i)} - \lambda^2_{2\,(i)}\lambda^2_{3\,(i)}\right)}} \\
& -\frac{1}{3}\,\frac{\sqrt{\frac{2}{3}\left(\lambda^4_{1\,(i)} + \lambda^4_{2\,(i)} + \lambda^4_{3\,(i)} - \lambda^2_{1\,(i)}\lambda^2_{2\,(i)} - \lambda^2_{1\,(i)}\lambda^2_{3\,(i)} - \lambda^2_{2\,(i)}\lambda^2_{3\,(i)}\right)}}{\lambda^2_{i\,(i)}} \Bigg)
\end{aligned} \tag{6.25}$$

since

$$\frac{\partial\left(\lambda^2_{1\,(i)}\lambda^2_{2\,(i)}\lambda^2_{3\,(i)}\right)}{\partial\lambda^2_{i\,(i)}} = \left(\delta_{1i}\lambda^2_{2\,(i)}\lambda^2_{3\,(i)} + \delta_{2i}\lambda^2_{1\,(i)}\lambda^2_{3\,(i)} + \delta_{3i}\lambda^2_{1\,(i)}\lambda^2_{2\,(i)}\right)$$

$$= \left(\lambda^2_{1\,(i)}\lambda^2_{2\,(i)}\lambda^2_{3\,(i)}\left(\frac{\delta_{1i}}{\lambda^2_{1\,(i)}} + \frac{\delta_{2i}}{\lambda^2_{2\,(i)}} + \frac{\delta_{3i}}{\lambda^2_{3\,(i)}}\right)\right) = J^{e\,2}\left(\frac{1}{\lambda^2_{i\,(i)}}\right) \tag{6.26}$$

$$\frac{\partial\lambda^2_{i\,(i)}}{\partial\Delta\gamma_{(i)}} = -2\,\bar{n}_{i\,(i)}exp(-2\Delta\gamma_{(i)}\bar{n}_{i\,(i)})\lambda^2_{i\,(trial)} \tag{6.27}$$

If these two tensors are contracted, a summation over the (i)-indices has to be carried out. Further tensors are :

$$\frac{\partial\lambda^2_{i\,(i)}}{\partial\bar{n}_{k\,(i)}} = (-2\Delta\gamma_{(i)}\delta_{ik})exp(-2\Delta\gamma_{(i)}\bar{n}_{i\,(i)})\lambda^2_{i\,(trial)} \tag{6.28}$$

$$\begin{aligned}\frac{\partial n_{k\,(i)}}{\partial\lambda^2_{i\,(i)}} &= \frac{\delta_{ik} - \frac{1}{3}\left(\delta_{1k}+\delta_{2k}+\delta_{3k}\right)}{\sqrt{\frac{2}{3}\left(\lambda^4_{1\,(i)}+\lambda^4_{2\,(i)}+\lambda^4_{3\,(i)}-\lambda^2_{1\,(i)}\lambda^2_{2\,(i)}-\lambda^2_{1\,(i)}\lambda^2_{3\,(i)}-\lambda^2_{2\,(i)}\lambda^2_{3\,(i)}\right)}} \\ &-\frac{\left(\lambda^2_{i\,(i)} - \frac{1}{3}(\lambda^2_{1\,(i)}+\lambda^2_{2\,(i)}+\lambda^2_{3\,(i)})\right)\left(\lambda^2_{k\,(i)} - \frac{1}{3}(\lambda^2_{1\,(i)}+\lambda^2_{2\,(i)}+\lambda^2_{3\,(i)})(\delta_{1k}+\delta_{2k}+\delta_{3k})\right)}{\left(\frac{2}{3}\left(\lambda^4_{1\,(i)}+\lambda^4_{2\,(i)}+\lambda^4_{3\,(i)}-\lambda^2_{1\,(i)}\lambda^2_{2\,(i)}-\lambda^2_{1\,(i)}\lambda^2_{3\,(i)}-\lambda^2_{2\,(i)}\lambda^2_{3\,(i)}\right)\right)^{3/2}} \\ &= \frac{\delta_{ik} - \frac{1}{3} - n_{i\,(i)}n_{k\,(i)}}{\sqrt{\frac{2}{3}\left(\lambda^4_{1\,(i)}+\lambda^4_{2\,(i)}+\lambda^4_{3\,(i)}-\lambda^2_{1\,(i)}\lambda^2_{2\,(i)}-\lambda^2_{1\,(i)}\lambda^2_{3\,(i)}-\lambda^2_{2\,(i)}\lambda^2_{3\,(i)}\right)}}\end{aligned} \tag{6.29}$$

where we may set :

$$(\delta_{1k}+\delta_{2k}+\delta_{3k}) = 1 \tag{6.30}$$

The local iteration is carried out until an error treshold is reached e.g. 10^{-12}.

6.4 The elasto-plastic tangent operator

6.4.1 Formulation in tensor notation

The inner virtual work reads as :

$$\delta W^{int} = \int \frac{\partial\psi^e}{\partial\bar{\mathbf{E}}} : \delta\bar{\mathbf{E}}\,dV = \int 2\,\frac{\partial\psi^e}{\partial\bar{\mathbf{C}}} : \frac{1}{2}\,\delta\bar{\mathbf{C}}\,dV = \int \mathbf{S}^{(n+1)} : \frac{1}{2}\,\delta\bar{\mathbf{C}}\,dV \tag{6.31}$$

If we now use quantities with respect to the current configuration, we have to use a so-called LIE-variation of $\bar{\mathbf{g}}$ ($\delta^b\bar{\mathbf{g}} = \mathbf{F}_\triangleright(\delta(\mathbf{F}^\triangleleft(\bar{\mathbf{g}})))$) :

$$\delta W^{int} = \int \boldsymbol{\tau}_{(n+1)} : \frac{1}{2}\,\delta^b\bar{\mathbf{g}}\,dV = \int \frac{1}{2}\,\delta^b\bar{\mathbf{g}}{\bullet\circ}\boldsymbol{\tau}_{(n+1)}dV \tag{6.32}$$

The first variation of δW^{int} linearized around the value $\delta W^{int} = 0$ reads as:

$$\Delta\delta W^{int} = \int \frac{1}{2}\Delta^b\delta^b\bar{\mathbf{g}}{\bullet\circ}\boldsymbol{\tau}_{(n+1)}\,dV + \int \frac{1}{2}\,\delta^b\bar{\mathbf{g}}{\bullet\circ}2\,\frac{\partial\boldsymbol{\tau}_{(n+1)}}{\partial\mathbf{g}}{\bullet\circ}\frac{1}{2}\,\Delta^b\bar{\mathbf{g}}\,dV \tag{6.33}$$

where we use the following dependency

$$\boldsymbol{\tau}_{(n+1)} = \boldsymbol{\tau}_{(n+1)}(\mathbf{g}, \mathbf{b}^e_{(n+1)}(\mathbf{n}_{(n+1)}, \Delta\gamma)) \tag{6.34}$$

resulting in :

$$\begin{aligned}\frac{\partial\boldsymbol{\tau}_{(n+1)}}{\partial\mathbf{g}} = {} & \frac{\partial\boldsymbol{\tau}_{(n+1)}}{\partial\mathbf{g}}\,|_{\mathbf{b}^e_{(n+1)}=const.} + \frac{\partial\boldsymbol{\tau}_{(n+1)}}{\partial\mathbf{b}^e_{(n+1)}}{\bullet\circ}\frac{\partial\mathbf{b}^e_{(n+1)}}{\partial\mathbf{n}_{(n+1)}}{\bullet\circ}\frac{\partial\mathbf{n}_{(n+1)}}{\partial\mathbf{g}} \\ & + \frac{\partial\boldsymbol{\tau}_{(n+1)}}{\partial\mathbf{b}^e_{(n+1)}}{\bullet\circ}\left(\frac{\partial\mathbf{b}^e_{(n+1)}}{\partial\Delta\gamma} \times \frac{\partial\Delta\gamma}{\partial\mathbf{g}}\right)\end{aligned} \tag{6.35}$$

Note that we have simply written $\mathbf{g}$ instead of the correct notation $\bar{\mathbf{g}} = \mathbf{g} + \tilde{\mathbf{g}}$. This will also be done in what follows. With the abbreviation :

$$\mathbb{E} = \left(\mathbf{i}\otimes\mathbf{i} + \frac{\partial\mathbf{R}_{\mathbf{n}\,(n+1)}}{\partial(\mathbf{g}\boldsymbol{\tau})_{(n+1)}}{\bullet\circ}\frac{\partial(\mathbf{g}\boldsymbol{\tau})_{(n+1)}}{\partial\mathbf{b}^e_{(n+1)}}{\bullet\circ}\frac{\mathbf{b}^e_{(n+1)}}{\partial\mathbf{n}_{(n+1)}}\right) \tag{6.36}$$

we obtain for the differentiation of the normal with respect to $\mathbf{g}$:

$$\begin{aligned}\frac{\partial\mathbf{n}_{(n+1)}}{\partial\mathbf{g}} = (-\mathbb{E}^{-1}){\bullet\circ}\; & \left[\frac{\partial\mathbf{R}_{\mathbf{n}\,(n+1)}}{\partial(\mathbf{g}\boldsymbol{\tau})_{(n+1)}}{\bullet\circ}\frac{\partial(\mathbf{g}\boldsymbol{\tau})_{(n+1)}}{\partial\mathbf{g}}\right. \\ & \left. + \frac{\partial\mathbf{R}_{\mathbf{n}\,(n+1)}}{\partial(\mathbf{g}\boldsymbol{\tau})_{(n+1)}}{\bullet\circ}\frac{\partial(\mathbf{g}\boldsymbol{\tau})_{(n+1)}}{\partial\mathbf{b}^e_{(n+1)}}{\bullet\circ}\left(\frac{\mathbf{b}^e_{(n+1)}}{\partial\Delta\gamma} \times \frac{\partial\Delta\gamma}{\partial\mathbf{g}}\right)\right]\end{aligned} \tag{6.37}$$

If we now introduce this term in the consistency condition ($\dot{F} = 0$):

$$\begin{aligned}& \frac{\partial F_{(n+1)}}{\partial(\mathbf{g}\boldsymbol{\tau})_{(n+1)}}{\bullet\circ}\frac{\partial(\mathbf{g}\boldsymbol{\tau})_{(n+1)}}{\partial\mathbf{g}} + \frac{\partial F_{(n+1)}}{\partial(\mathbf{g}\boldsymbol{\tau})_{(n+1)}}{\bullet\circ}\frac{\partial(\mathbf{g}\boldsymbol{\tau})_{(n+1)}}{\partial\mathbf{b}^e_{(n+1)}}{\bullet\circ}\left(\frac{\mathbf{b}^e_{(n+1)}}{\partial\Delta\gamma} \times \frac{\partial\Delta\gamma}{\partial\mathbf{g}}\right) \\ & + \frac{\partial F_{(n+1)}}{\partial(\mathbf{g}\boldsymbol{\tau})_{(n+1)}}{\bullet\circ}\frac{\partial(\mathbf{g}\boldsymbol{\tau})_{(n+1)}}{\partial\mathbf{b}^e_{(n+1)}}{\bullet\circ}\frac{\mathbf{b}^e_{(n+1)}}{\partial\mathbf{n}_{(n+1)}}{\bullet\circ}\frac{\partial\mathbf{n}_{(n+1)}}{\partial\mathbf{g}} + \frac{\partial F_{(n+1)}}{\partial q_{(n+1)}}\frac{\partial q_{(n+1)}}{\partial\alpha_{(n+1)}}\frac{\partial\alpha_{(n+1)}}{\partial\Delta\gamma}\frac{\partial\Delta\gamma}{\partial\mathbf{g}} = 0\end{aligned} \tag{6.38}$$

we finally get the derivative of the consistency parameter:

$$
\begin{aligned}
\frac{\partial \Delta\gamma}{\partial \mathbf{g}} = & -\Bigg[\left(\frac{\partial F_{(n+1)}}{\partial (\mathbf{g}\boldsymbol{\tau})_{(n+1)}} \bullet\circ \frac{\partial (\mathbf{g}\boldsymbol{\tau})_{(n+1)}}{\partial \mathbf{b}^e_{(n+1)}} \bullet\circ \frac{\mathbf{b}^e_{(n+1)}}{\partial \mathbf{n}_{(n+1)}}\right) \\
& \bullet\circ(-\mathbb{E}^{-1})\bullet\circ \left(\frac{\partial \mathbf{R}_{\mathbf{n}\,(n+1)}}{\partial (\mathbf{g}\boldsymbol{\tau})_{(n+1)}} \bullet\circ \frac{\partial (\mathbf{g}\boldsymbol{\tau})_{(n+1)}}{\partial \mathbf{b}^e_{(n+1)}} \bullet\circ \frac{\mathbf{b}^e_{(n+1)}}{\partial \Delta\gamma}\right) \\
& + \frac{\partial F_{(n+1)}}{\partial q_{(n+1)}} \frac{\partial q_{(n+1)}}{\partial \alpha_{(n+1)}} \frac{\partial \alpha_{(n+1)}}{\partial \Delta\gamma} + \frac{\partial F_{(n+1)}}{\partial (\mathbf{g}\boldsymbol{\tau})_{(n+1)}} \bullet\circ \frac{\partial (\mathbf{g}\boldsymbol{\tau})_{(n+1)}}{\partial \mathbf{b}^e_{(n+1)}} \bullet\circ \frac{\mathbf{b}^e_{(n+1)}}{\partial \Delta\gamma}\Bigg]^{-1} \\
& \Bigg[\left(\frac{\partial F_{(n+1)}}{\partial (\mathbf{g}\boldsymbol{\tau})_{(n+1)}} \bullet\circ \frac{\partial (\mathbf{g}\boldsymbol{\tau})_{(n+1)}}{\partial \mathbf{b}^e_{(n+1)}} \bullet\circ \frac{\mathbf{b}^e_{(n+1)}}{\partial \mathbf{n}_{(n+1)}}\right) \\
& \bullet\circ(-\mathbb{E}^{-1})\bullet\circ \left(\frac{\partial \mathbf{R}_{\mathbf{n}\,(n+1)}}{\partial (\mathbf{g}\boldsymbol{\tau})_{(n+1)}} \bullet\circ \frac{\partial (\mathbf{g}\boldsymbol{\tau})_{(n+1)}}{\partial \mathbf{g}}\right) + \frac{\partial F_{(n+1)}}{\partial (\mathbf{g}\boldsymbol{\tau})_{(n+1)}} \bullet\circ \frac{\partial (\mathbf{g}\boldsymbol{\tau})_{(n+1)}}{\partial \mathbf{g}}\Bigg]
\end{aligned}
\tag{6.39}
$$

This result included in **Equation** (6.37) and afterwards in **Equation** (6.35) allows the determination of the elasto-plastic tangent operator. Note that these equations are implemented in direct form. If all tensors **Equation** (6.7) are known as well as the constitutive laws, this offers now particular problems. In this way we have the possibility to consider arbitrary constitutive, in this case isotropic, material laws.

6.4.2 Formulation in principal axes

To derive the elasto-plastic tangent operator in eigenvalues it is suitable to start from the corresponding representation in absolute tensor notation and to express these equations in eigenvalues. Since the solution in absolute tensor notation is known, this transformation is simple. In absolute tensor notation the following residuals exist :

$$
\begin{aligned}
&\mathbf{n}_{(n+1)}(\mathbf{g}) - \mathbf{n}_{(n+1)}(\mathbf{g}, \mathbf{b}^e_{(n+1)}(\Delta\gamma(\mathbf{g}), \mathbf{n}_{(n+1)}(\mathbf{g}))) = \mathbf{0} \\
&F_{(n+1)}(\mathbf{g}, \mathbf{b}^e_{(n+1)}(\Delta\gamma(\mathbf{g}), \mathbf{n}_{(n+1)}(\mathbf{g})), q_{(n+1)}(\alpha_{(n+1)}(\Delta\gamma_{(n+1)}(\mathbf{g})))) = 0
\end{aligned}
\tag{6.40}
$$

Conclusively, we obtain :

$$
\begin{aligned}
&n_{i\,(n+1)}(\mathbf{g}) - n_{i\,(n+1)}\Big(\mathbf{g}, \lambda^2_{i\,(n+1)}(\Delta\gamma_{(n+1)}(\mathbf{g}), n_{(n+1)}(\mathbf{g})\Big) = 0 \\
&F_{(n+1)}\Big(\mathbf{g}, \lambda^2_{i\,(n+1)}\left(\Delta\gamma_{(n+1)}(\mathbf{g}), n_{(n+1)}(\mathbf{g})\right), q_{(n+1)}(\alpha_{(n+1)}(\Delta\gamma_{(n+1)}(\mathbf{g})))\Big) = 0
\end{aligned}
\tag{6.41}
$$

The differentiation with respect to $\mathbf{g}$ leads to the following results :

$$
\frac{n_{i\,(n+1)}}{\partial \mathbf{g}} - \frac{\partial n_{i\,(n+1)}}{\partial \mathbf{g}} - \frac{\partial n_{i\,(n+1)}}{\partial \lambda^2_{m\,(n+1)}} \frac{\partial \lambda^2_{m\,(n+1)}}{\partial \Delta\gamma_{(n+1)}} \frac{\partial \Delta\gamma_{(n+1)}}{\partial \mathbf{g}} - \frac{\partial n_{i\,(n+1)}}{\partial \lambda^2_{m\,(n+1)}} \frac{\partial \lambda^2_{m\,(n+1)}}{\partial n_{k\,(n+1)}} \frac{\partial n_{k\,(n+1)}}{\partial \mathbf{g}} = \mathbf{0}
\tag{6.42}
$$

It finally follows :

$$\frac{\partial n_{k\,(n+1)}}{\partial \mathbf{g}} = \left(\delta_{ik} - \frac{\partial n_{i\,(n+1)}}{\partial \lambda^2_{m\,(n+1)}} \frac{\partial \lambda^2_{m\,(n+1)}}{\partial n_{k\,(n+1)}}\right)^{-1} \left(\frac{\partial n_{i\,(n+1)}}{\partial \mathbf{g}} + \frac{\partial n_{i\,(n+1)}}{\partial \lambda^2_{m\,(n+1)}} \frac{\partial \lambda^2_{m\,(n+1)}}{\partial \Delta\gamma} \frac{\partial \Delta\gamma}{\partial \mathbf{g}}\right) \tag{6.43}$$

For this expression we need the derivative $\frac{\partial n_{i\,(n+1)}}{\partial \mathbf{g}}$, which will be derived, at first, in absolute tensor notation and will then be expressed in eigenvalues.

$$\begin{aligned}
\frac{\partial \mathbf{n}_{(n+1)}}{\partial \mathbf{g}} &= \frac{\frac{1}{2}\,(\mathbf{b}^e \otimes \mathbf{i} + \mathbf{b}^e \boxtimes \mathbf{i}) - \frac{1}{3}\,\mathbf{i} \times \mathbf{b}^e}{\sqrt{\mathrm{tr}(\mathbf{g}\mathbf{b}^e\mathbf{g}\mathbf{b}^e) - \frac{1}{3}\,(\mathrm{tr}(\mathbf{g}\mathbf{b}^e))^2}} \\
&\quad - \frac{\left(\mathbf{b}^e\mathbf{g} - \frac{1}{3}\,\mathrm{tr}(\mathbf{g}\mathbf{b}^e)\,\mathbf{i}\right) \times \left(\mathbf{b}^e\mathbf{g}\mathbf{b}^e - \frac{1}{3}\,\mathrm{tr}(\mathbf{g}\mathbf{b}^e)\,\mathbf{b}^e\right)}{\left(tr(\mathbf{g}\mathbf{b}^e\mathbf{g}\mathbf{b}^e) - \frac{1}{3}\,(tr(\mathbf{g}\mathbf{b}^e))^2\right)^{3/2}} \\
&= \frac{\left(\frac{1}{2}\,\lambda^2_{o\,(n+1)}\,(\mathbf{m}_o \otimes \mathbf{m}_i + \mathbf{m}_o \boxtimes \mathbf{m}_i) - \frac{1}{3}\,\mathbf{m}_o \times \lambda^2_{i\,(n+1)}\,\mathbf{m}_i - n_o\,\mathbf{m}_o \times \lambda^2_{i\,(n+1)} n_i\,\mathbf{m}_i\right)}{\sqrt{\frac{2}{3}\,(\lambda_1^4 + \lambda_2^4 + \lambda_3^4 - \lambda_1^2\lambda_2^2 - \lambda_1^2\lambda_3^2 - \lambda_2^2\lambda_3^2)_{(n+1)}}}
\end{aligned} \tag{6.44}$$

where for a term $\mathbf{m}_o$ or $\mathbf{m}_i$ without prefactor with the same index the summation in the form $\mathbf{m}_o = \mathbf{m}_1 + \mathbf{m}_2 + \mathbf{m}_3$ has to be carried out. This rule has also to be followed in all relations which are given latter.

The contraction from the left-hand side with $\mathbf{g}^o \otimes \mathbf{g}_p$, i.e. $(\mathbf{g}^o \otimes \mathbf{g}_p){\cdot\circ}(\frac{\partial \mathbf{n}}{\partial \mathbf{g}})$ gives $\frac{\partial n^o_{.\,p}}{\partial \mathbf{g}}$. Finally, if we transform this into eigenvalues, we obtain :

$$\frac{\partial n_{k\,(n+1)}}{\partial \mathbf{g}} = \frac{\left(\lambda^2_{k\,(n+1)}\mathbf{m}_k - \lambda^2_{i\,(n+1)}(\frac{1}{3} + n_k\,n_i)\,\mathbf{m}_i\right)}{\sqrt{\frac{2}{3}\,(\lambda_1^4 + \lambda_2^4 + \lambda_3^4 - \lambda_1^2\lambda_2^2 - \lambda_1^2\lambda_3^2 - \lambda_2^2\lambda_3^2)_{(n+1)}}} \tag{6.45}$$

(no summation over k, but over i !)

The derivative of the yield function is given by :

$$\begin{aligned}
&\frac{\partial F_{(n+1)}}{\partial \mathbf{g}} + \frac{\partial F_{(n+1)}}{\partial \lambda^2_{i\,(n+1)}} \frac{\partial \lambda^2_{i\,(n+1)}}{\partial n_{k\,(n+1)}} \frac{\partial n_{k\,(n+1)}}{\partial \mathbf{g}} + \frac{\partial F_{(n+1)}}{\partial \lambda^2_{i\,(n+1)}} \frac{\partial \lambda^2_{i\,(n+1)}}{\partial \Delta\gamma_{(n+1)}} \frac{\partial \Delta\gamma_{(n+1)}}{\partial \mathbf{g}} \\
&+ \frac{\partial F_{(n+1)}}{\partial q_{(n+1)}} \frac{\partial q_{(n+1)}}{\partial \alpha_{(n+1)}} \frac{\partial \alpha_{(n+1)}}{\partial \Delta\gamma_{(n+1)}} \frac{\partial \Delta\gamma_{(n+1)}}{\partial \mathbf{g}} = \mathbf{0}
\end{aligned} \tag{6.46}$$

Using **Equation** (6.43) finally delivers :

$$\frac{\partial \Delta\gamma_{(n+1)}}{\partial \mathbf{g}} = -\left[\frac{\partial F_{(n+1)}}{\partial \lambda^2_{i\,(n+1)}}\frac{\partial \lambda^2_{i\,(n+1)}}{\partial \Delta\gamma_{(n+1)}} + \frac{\partial F_{(n+1)}}{\partial q_{(n+1)}}\frac{\partial q_{(n+1)}}{\partial \alpha_{(n+1)}}\frac{\partial \alpha_{(n+1)}}{\partial \Delta\gamma_{(n+1)}}\right.$$
$$\left. + \frac{\partial F_{(n+1)}}{\partial \lambda^2_{i\,(n+1)}}\frac{\partial \lambda^2_{i\,(n+1)}}{\partial n_{k\,(n+1)}}\left(\delta_{lk} - \frac{\partial n_{l\,(n+1)}}{\partial \lambda^2_{n\,(n+1)}}\frac{\partial \lambda^2_{n\,(n+1)}}{\partial n_{k\,(n+1)}}\right)^{-1}\frac{\partial n_{l\,(n+1)}}{\partial \lambda^2_{m\,(n+1)}}\frac{\partial \lambda^2_{m\,(n+1)}}{\partial \Delta\gamma_{(n+1)}}\right]^{-1}$$
$$\left[\frac{\partial F_{(n+1)}}{\partial \lambda^2_{i\,(n+1)}}\frac{\partial \lambda^2_{i\,(n+1)}}{\partial n_{k\,(n+1)}}\left(\delta_{lk} - \frac{\partial n_{l\,(n+1)}}{\partial \lambda^2_{n\,(n+1)}}\frac{\partial \lambda^2_{n\,(n+1)}}{\partial n_{k\,(n+1)}}\right)^{-1}\frac{\partial n_{l\,(n+1)}}{\partial \mathbf{g}} + \frac{\partial F_{(n+1)}}{\partial \mathbf{g}}\right] \tag{6.47}$$

with

$$\frac{\partial F_{(n+1)}}{\partial \mathbf{g}} = \mu\, J^{e\,-2/3}\left(\frac{(\mathbf{b}^e\mathbf{g}\mathbf{b}^e - \frac{1}{3}\,\mathrm{tr}(\mathbf{g}\mathbf{b}^e)\mathbf{b}^e)}{\sqrt{\left(\mathrm{tr}(\mathbf{g}\mathbf{b}^e\mathbf{g}\mathbf{b}^e) - \frac{1}{3}(tr(\mathbf{g}\mathbf{b}^e))^2\right)}} - \frac{1}{3}\sqrt{\left(\mathrm{tr}(\mathbf{g}\mathbf{b}^e\mathbf{g}\mathbf{b}^e) - \frac{1}{3}(\mathrm{tr}(\mathbf{g}\mathbf{b}^e))^2\right)}\;\mathbf{g}^{-1}\right)$$
$$= \sum_{i=1}^{3}\mu\, J^{e\,-2/3}\left(\frac{\lambda^4_{i\,(n+1)} - \frac{1}{3}\,(\lambda_1^2+\lambda_2^2+\lambda_3^2)^{(n+1)}\lambda^2_{i\,(n+1)}}{\sqrt{\frac{2}{3}\,(\lambda_1^4+\lambda_2^4+\lambda_3^4-\lambda_1^2\lambda_2^2-\lambda_1^2\lambda_3^2-\lambda_2^2\lambda_3^2)_{(n+1)}}}\right.$$
$$\left. - \frac{1}{3}\sqrt{\frac{2}{3}\,(\lambda_1^4+\lambda_2^4+\lambda_3^4-\lambda_1^2\lambda_2^2-\lambda_1^2\lambda_3^2-\lambda_2^2\lambda_3^2)_{(n+1)}}\right)\mathbf{m}_i \tag{6.48}$$

Now the elasto-plastic tangent operator will be derived. The dependency is :

$$\boldsymbol{\tau}_{(n+1)}(\mathbf{g}, \mathbf{b}^e_{(n+1)}(\Delta\gamma_{(n+1)}(\mathbf{g}), \mathbf{n}_{(n+1)}(\mathbf{g}))) \tag{6.49}$$

Since the eigenprojection is invariant :

$$\boldsymbol{\tau}_{(n+1)}(\mathbf{g}, \lambda^2_{i\,(n+1)}(\Delta\gamma_{(n+1)}(\mathbf{g}), n_{m\,(n+1)}(\mathbf{g}))) \tag{6.50}$$

$$\frac{\partial \boldsymbol{\tau}_{(n+1)}}{\partial \mathbf{g}} = \frac{\partial \boldsymbol{\tau}_{(n+1)}}{\partial \mathbf{g}}\,|_{(\Delta\gamma, n_m)=const.} + \frac{\partial \boldsymbol{\tau}_{(n+1)}}{\partial \lambda^2_{i\,(n+1)}}\frac{\partial \lambda^2_{i\,(n+1)}}{\partial \Delta\gamma_{(n+1)}} \times \frac{\partial \Delta\gamma_{(n+1)}}{\partial \mathbf{g}}$$
$$\underline{+ \frac{\partial \boldsymbol{\tau}_{(n+1)}}{\partial \lambda^2_{i\,(n+1)}}\frac{\partial \lambda^2_{i\,(n+1)}}{\partial n_{k\,(n+1)}} \times \frac{\partial n_{k\,(n+1)}}{\partial \mathbf{g}}} \tag{6.51}$$

It has to be noted that the computation of the elasto-plastic tangent operator in the above way is not fully correct. Like it can be seen, the underlined terms are reduced to a tensor product of the form ($\times$). Thereby the exact solution for the real fourth-order tensor is somehow constrained which is not correct. If we compare **Equation** (6.50) with

Equation (6.35) we see that this conclusion does not hold for the term involving $\Delta\gamma_{,\mathbf{g}}$. However, the underlined term has to be computed alternatively. The following fourth-order tensor has to be transformed into eigenvalues (see **Equations** (6.35), (6.36) and (6.37)) :

$$\frac{\partial \boldsymbol{\tau}^{(n+1)}}{\partial \mathbf{b}^e} \bullet\circ \frac{\partial \mathbf{b}^e}{\partial \mathbf{n}} \bullet\circ \left(\mathbb{I} - \frac{\partial \mathbf{n}}{\partial \mathbf{b}^e} \frac{\partial \mathbf{b}^e}{\partial \bar{\mathbf{n}}} \right)^{-1} \bullet\circ \left(\frac{\partial \mathbf{n}}{\partial \mathbf{g}} + \frac{\partial \mathbf{n}}{\partial \mathbf{b}^e} \bullet\circ \left(\frac{\partial \mathbf{b}^e}{\partial \Delta\gamma} \times \frac{\partial \Delta\gamma}{\partial \mathbf{g}} \right) \right) \tag{6.52}$$

However, also in this case we can employ a simplification. Instead of inserting a 9×9-matrix in the above middle term, only the inversion of a 3×3- and of three 2×2-matrices is necessary. This yields to :

$$\left(\mathbb{I} - \frac{\partial \mathbf{n}}{\partial \mathbf{b}^e} \frac{\partial \bar{\mathbf{b}}^e}{\partial \bar{\mathbf{n}}} \right) = \left(\begin{array}{c|c|c|c} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \mathbf{A}_1 & 0 & 0 & 0 \\ \hline 0 & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \mathbf{A}_2 & 0 & 0 \\ \hline 0 & 0 & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \mathbf{A}_3 & 0 \\ \hline 0 & 0 & 0 & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \mathbf{A}_4 \end{array} \right) \tag{6.53}$$

with

$$\mathbf{A}_1 = \left(\begin{array}{c|c|c} Exp_1^2\ \lambda_1\ \Delta\gamma \ \dfrac{(-4/3 + 2\, n_1^2)}{\sqrt{\cdots}} & Exp_2^2\ \lambda_2\ \Delta\gamma \ \dfrac{(2/3 + 2\, n^{12})}{\sqrt{\cdots}} & Exp_3^2\ \lambda_3\ \Delta\gamma \ \dfrac{(2/3 + 2\, n^{13})}{\sqrt{\cdots}} \\ \hline Exp_1^2\ \lambda_1\ \Delta\gamma \ \dfrac{(2/3 + 2\, n^{12})}{\sqrt{\cdots}} & Exp_2^2\ \lambda_2\ \Delta\gamma \ \dfrac{(-4/3 + 2\, n_2^2)}{\sqrt{\cdots}} & Exp_3^2\ \lambda_3\ \Delta\gamma \ \dfrac{(2/3 + 2\, n^{23})}{\sqrt{\cdots}} \\ \hline Exp_1^2\ \lambda_1\ \Delta\gamma \ \dfrac{(2/3 + 2\, n^{13})}{\sqrt{\cdots}} & Exp_2^2\ \lambda_2\ \Delta\gamma \ \dfrac{(2/3 + 2\, n^{23})}{\sqrt{\cdots}} & Exp_3^2\ \lambda_3\ \Delta\gamma \ \dfrac{(-4/3 + 2\, n_3^2)}{\sqrt{\cdots}} \end{array} \right) \tag{6.54}$$

and

$$\mathbf{A}_2 = n_{12} \left(\begin{array}{c|c} Exp_2\, \lambda_2 & Exp_1\, \lambda_1 \\ Exp_2\, \lambda_2 & Exp_1\, \lambda_1 \end{array} \right) \qquad \mathbf{A}_3 = n_{23} \left(\begin{array}{c|c} Exp_3\, \lambda_3 & Exp_2\, \lambda_2 \\ Exp_3\, \lambda_3 & Exp_2\, \lambda_2 \end{array} \right) \tag{6.55}$$

$$\mathbf{A}_4 = n_{13} \left(\begin{array}{c|c} Exp_3\, \lambda_3 & Exp_1\, \lambda_1 \\ Exp_3\, \lambda_3 & Exp_1\, \lambda_1 \end{array} \right) \tag{6.56}$$

The order of components of the second-order tensor in the above matrix is defined by :

$$\left(\begin{array}{c|c|c|c|c|c|c|c|c} 11 & 22 & 33 & 12 & 21 & 23 & 32 & 13 & 31 \end{array} \right) \tag{6.57}$$

We will use the following abbreviations (with $n_a = n_{a\,(n+1)}$) :

$$\begin{array}{c} Exp_a = \exp(-\Delta\gamma\, n_a) \qquad Exp_a^2 = \exp(-2\,\Delta\gamma\, n_a) \qquad \lambda_a = \lambda^2_{a\,(trial)} \\ n^{ab} = n^{ba} = n_a\, n_b \qquad n_a^2 = n_a\, n_a \qquad n_{ab} = n_{ba} = \sum_{k=1}^{\infty} \sum_{r=0}^{k-1} \frac{(-\Delta\gamma)^k}{k!}\, n_b^r\, n_a^{k-r-1} \end{array} \tag{6.58}$$

A further important quantity is obtained form the differentiation of relation **Equation** (5.73) in the form :

$$\frac{\partial \boldsymbol{\tau}_{(n+1)}}{\partial \lambda^2_{k\,(n+1)}} = \left(\mu\, J^{e\,-2/3}\right) \mathbf{m}_k + \left(-\frac{1}{3}\,\mu\, J^{e\,-2/3} + \frac{\left(\frac{1}{2}\,\kappa - \frac{1}{3}\,\mu\, J^{e\,-2/3}\left(\lambda_i^2 - \frac{1}{3}(\lambda_1^2+\lambda_2^2+\lambda_3^2)\right)_{(n+1)}\right)}{\lambda^2_{k\,(n+1)}} \right) \mathbf{m}_i \tag{6.59}$$

$$\begin{aligned} \frac{\partial \boldsymbol{\tau}_{(n+1)}}{\partial \mathbf{b}^e} = {} & \frac{1}{2}\,\kappa\, \mathbf{m}_o \times \frac{1}{\lambda^2_{i\,(n+1)}}\mathbf{m}_i + \mu\, J^{e\,-2/3} \left(-\frac{1}{3}\lambda^2_{o\,(n+1)}\, \mathbf{m}_o \times \frac{1}{\lambda_i^{2\,(n+1)}}\mathbf{m}_i - \frac{1}{3}\,\mathbf{m}_o \times \mathbf{m}_i \right. \\ & \left. + \frac{1}{9}\,(\lambda_1^2+\lambda_2^2+\lambda_3^2)_{(n+1)}\, \mathbf{m}_o \times \frac{1}{\lambda^2_{i\,(n+1)}}\,\mathbf{m}_i + \frac{1}{2}(\mathbf{m}_o \otimes \mathbf{m}_i + \mathbf{m}_o \boxtimes \mathbf{m}_i) \right) \end{aligned} \tag{6.60}$$

and

$$\frac{\partial \mathbf{n}}{\partial \mathbf{b}^e} = \frac{\left(\frac{1}{2}(\mathbf{m}_o \otimes \mathbf{m}_i + \mathbf{m}_o \boxtimes \mathbf{m}_i) - \frac{1}{3}\,\mathbf{m}_o \times \mathbf{m}_i - n_o\,\mathbf{m}_o \times n_i\,\mathbf{m}_i\right)}{\sqrt{\frac{2}{3}\,(\lambda_1^4+\lambda_2^4+\lambda_3^4-\lambda_1^2\lambda_2^2-\lambda_1^2\lambda_3^2-\lambda_2^2\lambda_3^2)_{(n+1)}}} \tag{6.61}$$

as well as

$$\frac{\partial \mathbf{b}^e}{\partial \Delta\gamma} = -2\, n_i\, \lambda^2_{i\,(trial)}\, \exp(-2\Delta\gamma n_i) \tag{6.62}$$

The derivative of $\mathbf{b}^e$ with respect to $\mathbf{n}$ can be decisively simplified, by using the matrices ($\mathbf{A}_i\,(i=2,3,4)$) as before :

$$\frac{\partial \mathbf{b}^e}{\partial \mathbf{n}} = \left(\begin{array}{c|c|c|c} (-2\,\Delta\gamma) \left(\begin{array}{c|c|c} Exp_1^2\,\lambda_1 & 0 & 0 \\ \hline 0 & Exp_2^2\,\lambda_2 & 0 \\ \hline 0 & 0 & Exp_3^2\,\lambda_3 \end{array} \right) & 0 & 0 & 0 \\ \hline 0 & \mathbf{A}_2 & 0 & 0 \\ \hline 0 & 0 & \mathbf{A}_3 & 0 \\ \hline 0 & 0 & 0 & \mathbf{A}_4 \end{array} \right) \tag{6.63}$$

with the above given order **Equation** (6.57) for the components of a second-order tensor. Finally we sort the above terms into the corresponding positions of a 4-order tensor. To compute the elasto-plastic tangent, now only the elastic part is missing, which can be obtained as follows (see **Equation** (5.69)):

$$\begin{aligned} \frac{\partial \boldsymbol{\tau}_{(n+1)}}{\partial \mathbf{g}} \,|_{(\cdots)=const.} = \; & \left(\kappa + \frac{2}{3}\,\mu\, J^{e\,-2/3} \left(-\lambda^2_{i\,(n+1)} - \lambda^2_{k\,(n+1)} + \frac{1}{3}(\lambda_1^2+\lambda_2^2+\lambda_3^2)_{(n+1)} \right) \right) \mathbf{m}_i \times \mathbf{m}_k \\ & + \left(-\kappa\, \ln J^e + \frac{1}{3}\,\mu\, J^{e\,-2/3}\,(\lambda_1^2+\lambda_2^2+\lambda_3^2)_{(n+1)} \right) (\mathbf{m}_i \otimes \mathbf{m}_k + \mathbf{m}_i \boxtimes \mathbf{m}_k) \end{aligned} \tag{6.64}$$

In all equations the determinant $J^e = \sqrt{\lambda_1^2\lambda_2^2\lambda_3^2}$ is evaluated at the specific point, here at the end of the iteration $(n+1)$.

Chapter 7

Numerical examples

7.1 Simple shear test

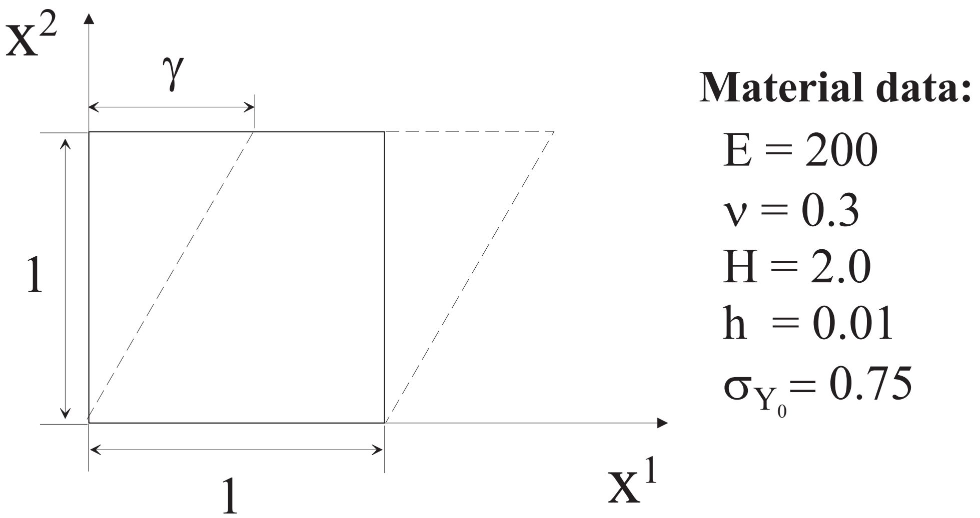

Figure **7.1**: Simple shear of a rectangular sheet (plane strain analysis)

As first example simple shear of a rectangular sheet assuming plane strain conditions is considered. **Figure** (7.1) contains the geometry and material parameters. In **Figure** (7.2)-(7.5) the CAUCHY-stresses $\sigma^{<11>}$ and $\sigma^{<12>}$ are plotted versus the shear strain γ for different models, which are tensor model and model in principal axes . For these implicit models smooth results are obtained from the model algorithm formulated in the tensor notation and in the principal axes notation (eigenvalues notation) . The results for the different models are in good agreement with each other. In case of isotropic hardening the elasto-plastic tangent operator possesses major and minor symmetry.

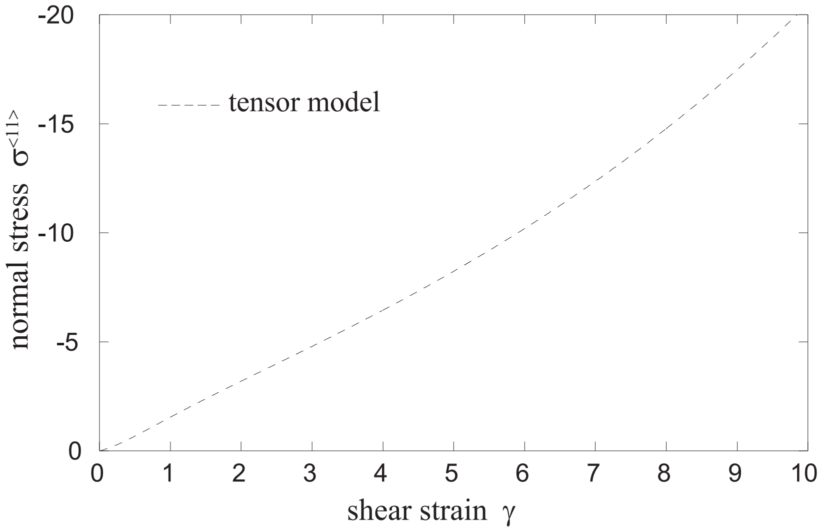

Figure **7.2**: Simple shear of a rectangular sheet (isotropic hardening) CAUCHY-stress versus shear strain of tensor model

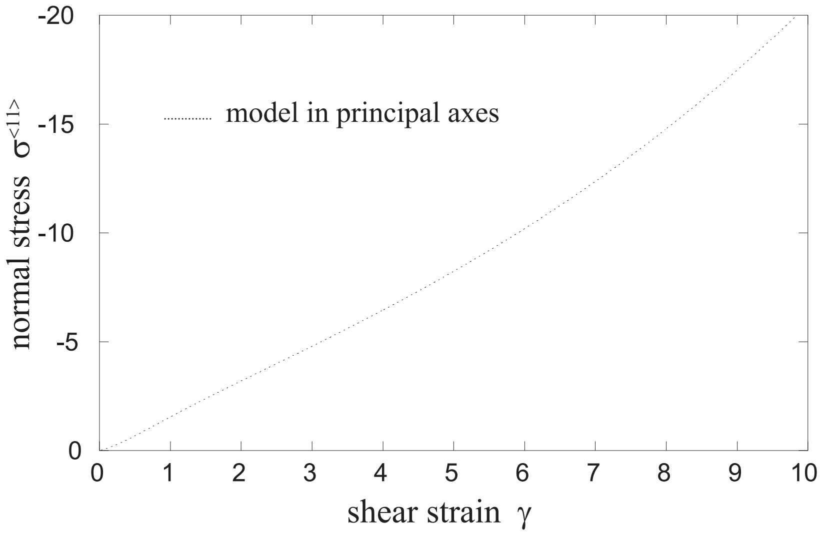

Figure **7.3**: Simple shear of a rectangular sheet (isotropic hardening) CAUCHY-stress versus shear strain of model in principal axes

The CAUCHY-stresses versus shear strain results are compared between the tensor model and model in principal axes, in **Table 7.1**.

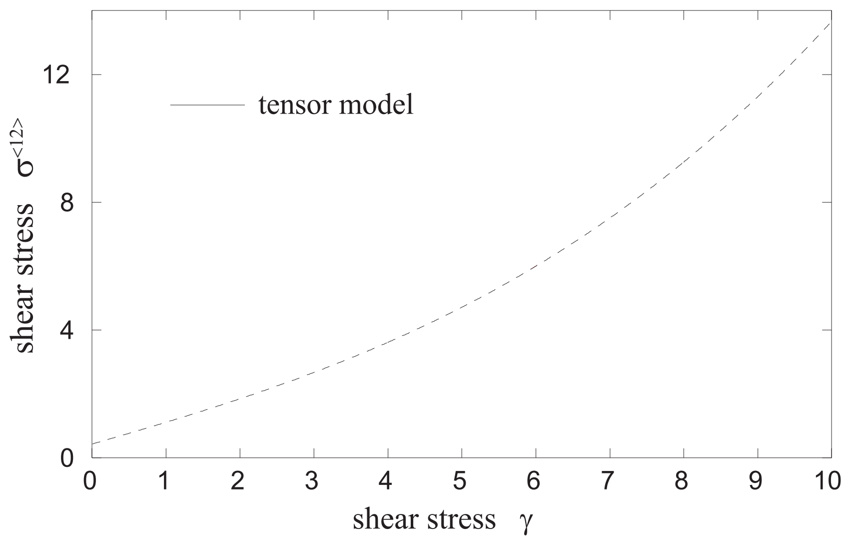

Figure **7.4**: Simple shear of a rectangular sheet (isotropic hardening) CAUCHY-stress versus shear strain of tensor model

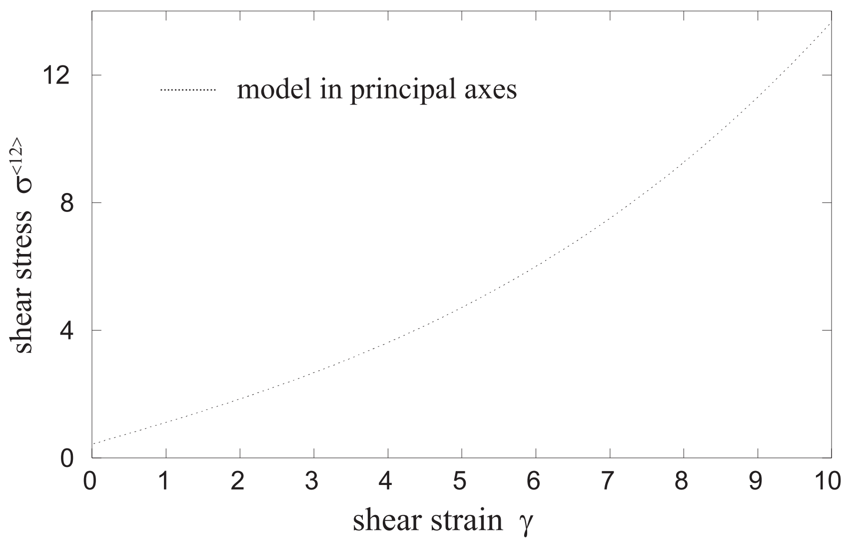

Figure **7.5**: Simple shear of a rectangular sheet (isotropic hardening) CAUCHY-stress versus shear strain of model in principal axes

7.2 Tension test

As next example we consider a single element under uniaxial tension. **Figure (7.6)** contains the geometry and material parameters. In **Figures** (7.7) and (7.8) the CAUCHY-stress $\sigma^{<22>}$ is plotted versus displacements [U], which are prescribed at the top, for both models. For these two models the results are very identical to each other.

The CAUCHY-stress $\sigma^{<22>}$ versus dispalcements [U] results are compared between

Table **7.1**: Comparison between the tensor model and model in principal axes for the simple shear test

	Shear strain γ	Tensor model	Model in pricipal axes
Normal stress $\sigma^{<11>}$	$.10006 \times 10^2$	-0.20511×10^2	-0.20511×10^2
Shear stress $\sigma^{<12>}$	0.56291×10^{-2}	0.4331	0.4331
	$.10006 \times 10^2$	0.13681×10^2	0.13681×10^2

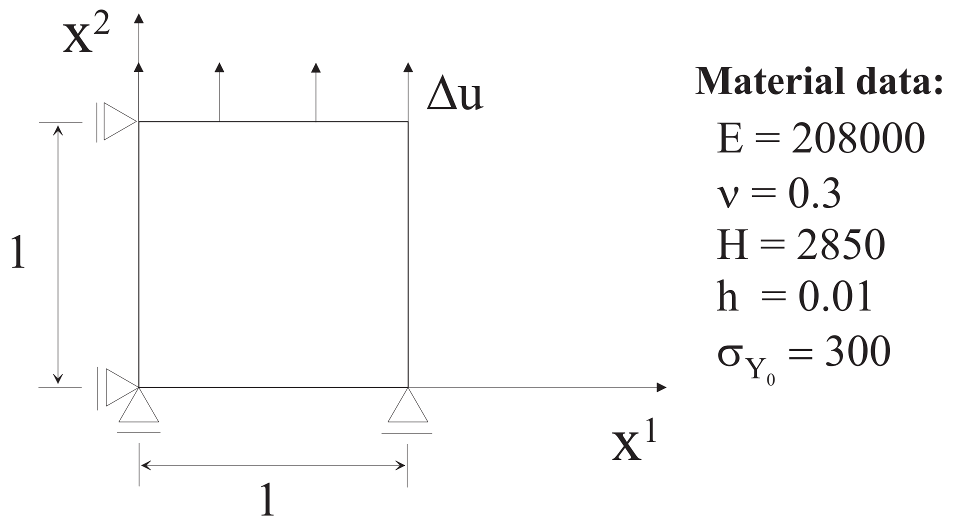

Figure **7.6**: Tension of a rectangular sheet

the tensor model and model in principal axes, in **Table 7.2**.

7.3 Perforated strip under uniaxial extension

A thin perforated strip under uniaxial extension is considered in this example. This example serves as a good test benchmark to test the proper working of the EAS-concept, since without EAS we would obtain no decline in **F** after the maximum has been reached. Due to symmetry of the structure only one quarter has been discretized (for details concerning the geometry, material parameters and discretization using 60 enhanced-strain 4-noded elements see **Figure** (7.9)). The computation has been carried out by prescribing displacements Δu at the right-hand side of the structure. The reaction force **F** at the left-hand side has been recorded. As can be observed in the load-displacement diagrams **Figure** (7.10) and (7.11) for both models the results are identical.

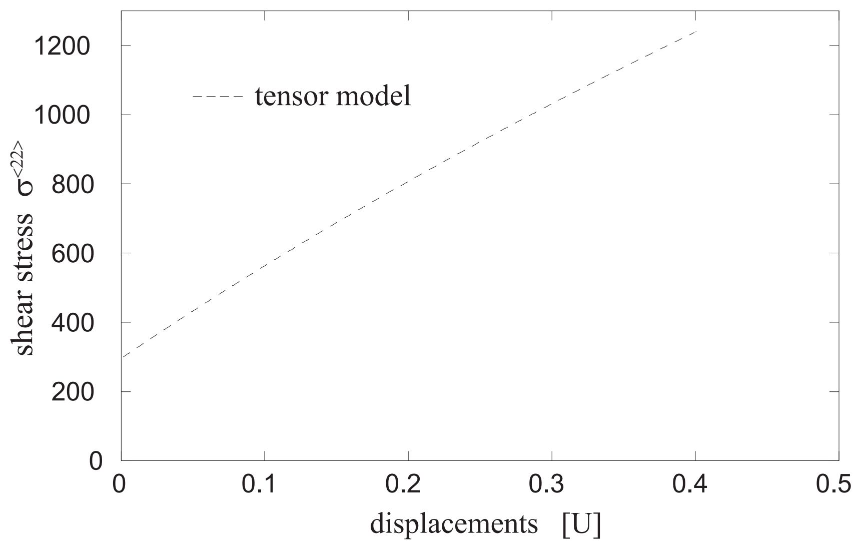

Figure **7.7**: Cauchy-stress versus displacements of tensor model

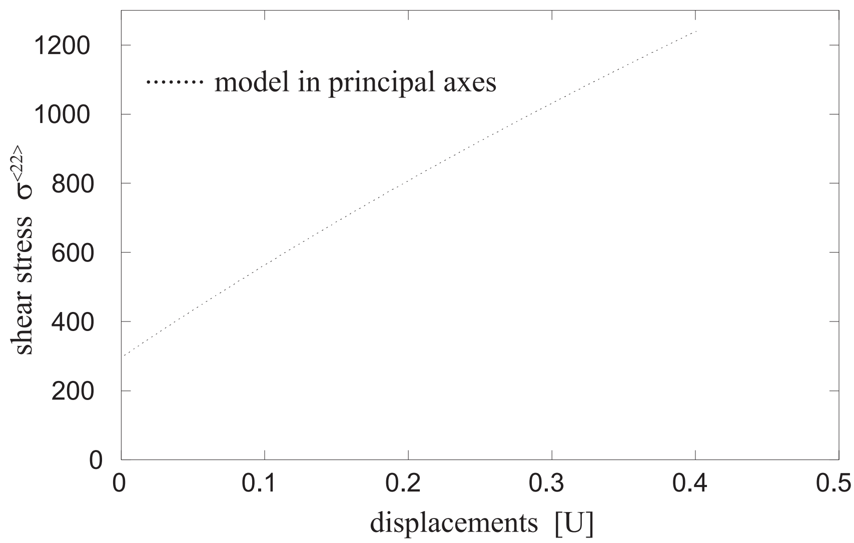

Figure **7.8**: Cauchy-stress versus displacements of model in principal axes

The force **F** versus dispalcements Δu results are compared between the tensor model and model in principal axes, in **Table 7.3**.

Table **7.2**: Comparison between the tensor model and model in principal axes of tension test

	Displacements [U]	Tensor model	Model in pricipal axes
Shear stress $\sigma^{<22>}$	$.14421 \times 10^{-2}$	0.29974×10^{3}	0.29974×10^{3}
	.40044	0.12397×10^{4}	0.12397×10^{4}

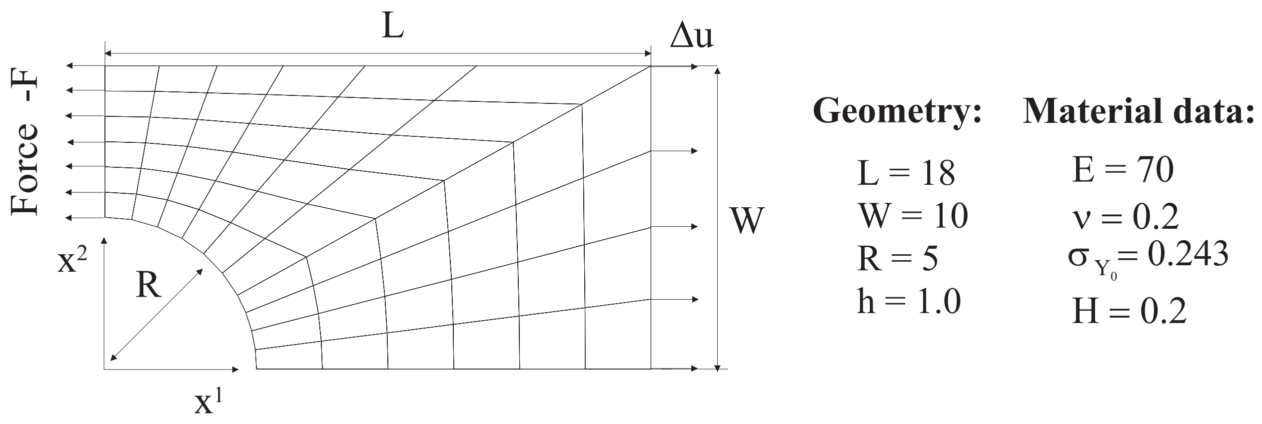

Figure **7.9**: Perforated strip under uniaxial extension geometry and finite element mesh

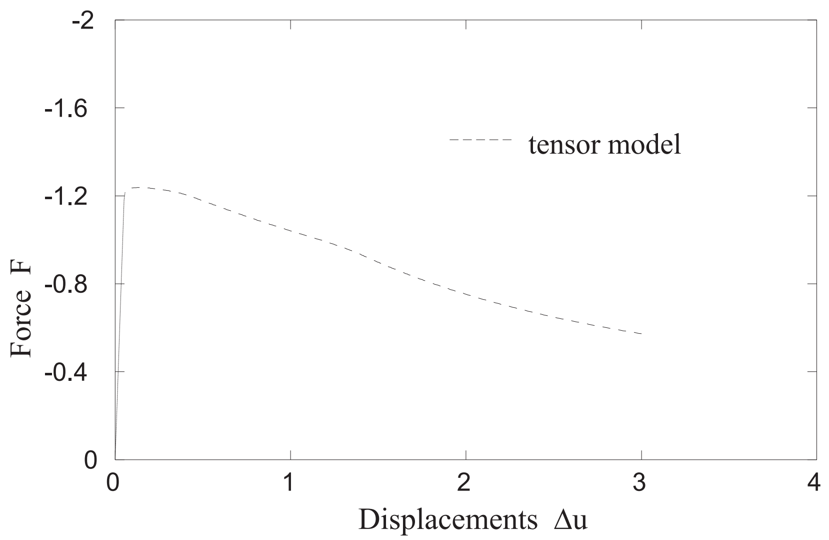

Figure **7.10**: Load-displacement curve of tensor model (isotropic hardening)

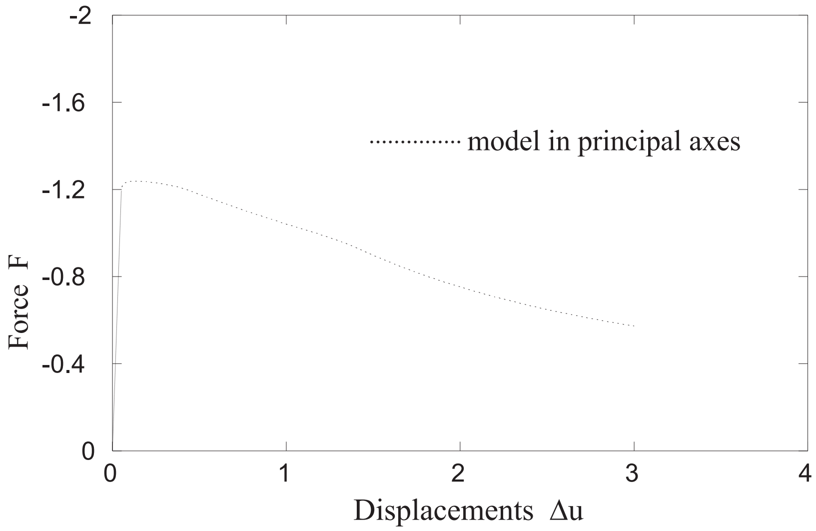

Figure **7.11**: Load-displacement curve of model in principal axes (isotropic hardening)

Table **7.3**: Comparison between the tensor model and model in principal axes of perforated strip under uniaxial extension

	Displacements Δu	Tensor model	Model in pricipal axes
Force **F**	0.135	-0.12380×10^1	-0.12380×10^1
	1.0	-0.10413×10^1	-0.10413×10^1
	2.0	-0.75255	-0.75255
	3.0	-0.57308	-0.57308

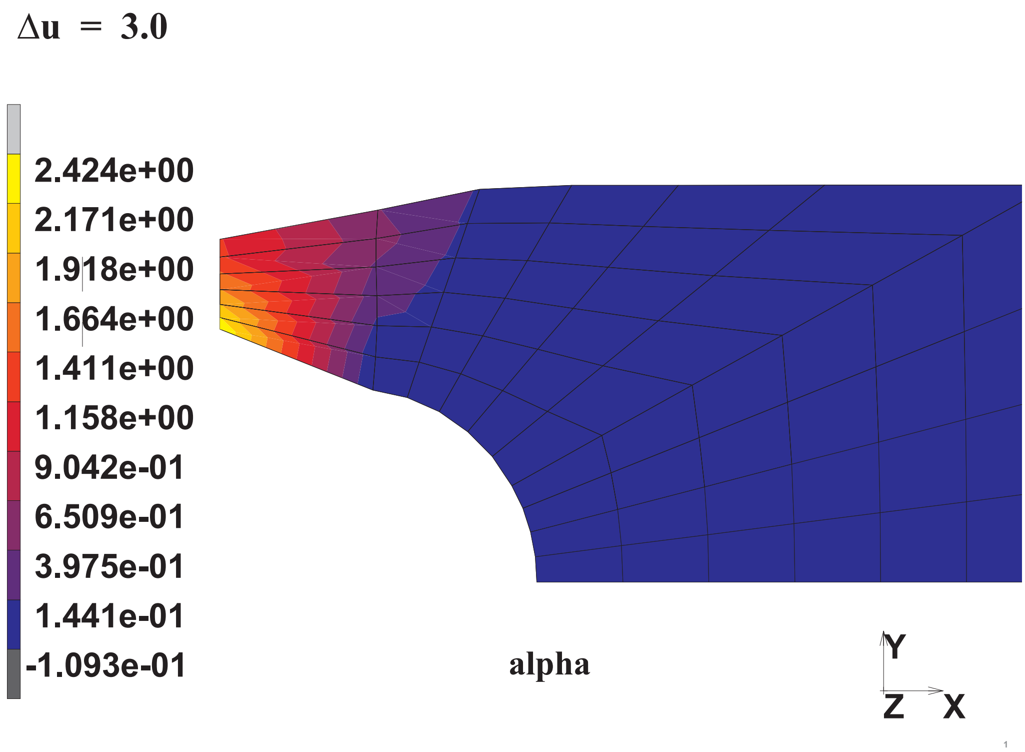

Figure **7.12**: Equivalent plastic strain alpha α of perforated strip (isotropic hardening) of tensor model

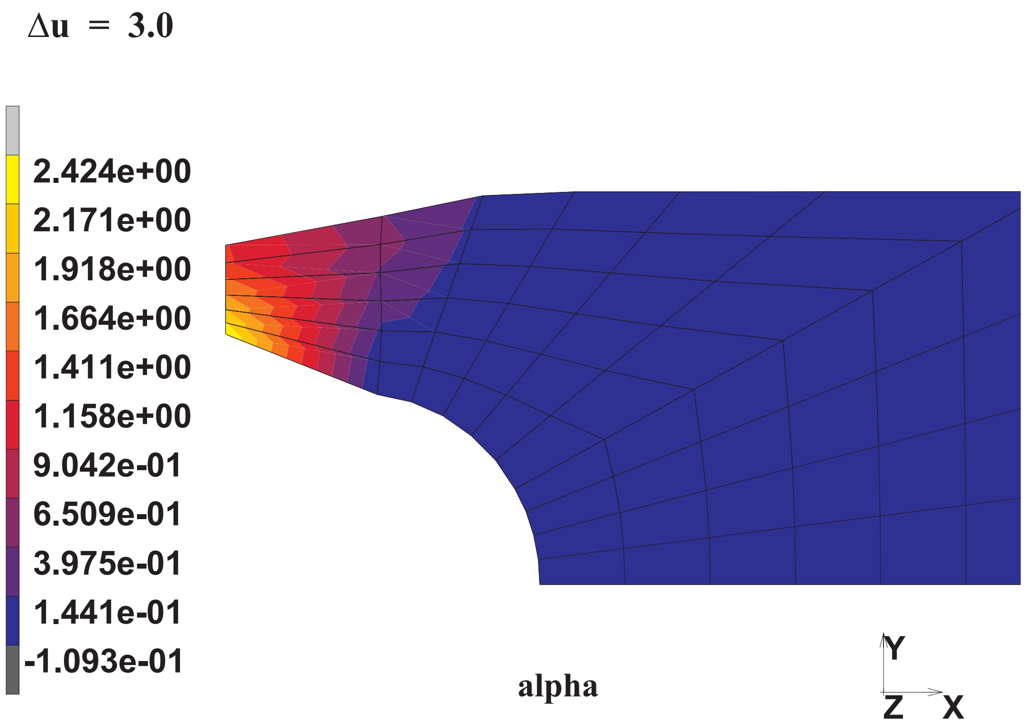

Figure **7.13**: Equivalent plastic strain alpha α of perforated strip (isotropic hardening) of model in principal axes

Chapter 8

Conclusions

The main aim of this work is to compare the isotropic plastic model formulated in eigenvales and in tensor formulation.

Form this study the following conclusions are drawn.

- The dimensionality of the isotropic plastic model in eigenvalues is reduced from 10 to 4.
- We could have reduced the tensor related problem also to a 7-dimensional problem by considering a normal $F_{,\mathbf{b}^e\mathbf{g}\boldsymbol{\tau}}$ instead of $F_{,\mathbf{g}\boldsymbol{\tau}}$ in the residual, since the components of $\mathbf{b}^e\mathbf{g}\boldsymbol{\tau}$ are for isotropic hyperelasticity always symmetric.
- The eigenprojections of $\mathbf{b}^e$ are invariant during the Return map. This leads to a the 4-dimensional problem, where the consistency parameter and the three eigenvalues of the normal to the yield surface are independent variables.
- The model relies on the isotropy of hyperelastic law, $\mathbf{b}^e$ as driving variable, the coaxiality of the KIRCHOFF-stress measure $\boldsymbol{\tau}$ and $\mathbf{b}^e$, the usual VON MISES-J_2-yield function and the evolution of $\mathbf{b}^e$ by means of an exponential map.
- The results obtained in the numerical examples for the model formulated in tensor notation and the model formulated in the principal axes are identical.
- Return Map in eigenvalues is similar to the Return Map in tensor formulation with the exception that we now can simplify a lot which saves computational cost.
- The elasto-pastic tangent operator is used in computations of the global solutions.

Chapter 9

Appendix

9.1 Computation of the eigenprojections

Before the Return Map, the eigenvalues and eigenprojections have to be computed (we set $\bar{\lambda} = \lambda^2$ for the eigenvalues of $\mathbf{b}^e$). To treat multiple eigenvalues we can exploit the essential fact that all eigenprojections are invariant during the Return Map. This allows the implementation of just one unified algorithm for three eigenvalues by considering the special cases by means of certain prefactors which are used in the computation of the final value of $\mathbf{b}^e$ after the Return Map is accomplished.

- Treatment of equal eigenvalues is done as follows:
  Let $\Delta\bar{\lambda}_1 = (\bar{\lambda}_1 - \bar{\lambda}_2)$, $\Delta\bar{\lambda}_2 = (\bar{\lambda}_2 - \bar{\lambda}_3)$ and $\Delta\bar{\lambda}_3 = (\bar{\lambda}_1 - \bar{\lambda}_3)$ where the $\bar{\lambda}_i$ are the eigenvalues of the $\mathbf{b}^e$. If three eigenvalues are equal then icase = 3, if two eigenvalues are equal then icase = 2 and if all three eigenvalues are different then icase = 1. Here $Tol = 1.0\,\mathrm{e}^{-12}$. The algorithm is given as follows:

```
If (Δλ̄1 < Tol)   and   (Δλ̄2 < Tol)    then
icase = 3
elseif (Δλ̄3 < Tol)    then
icase = 2
λ̄ = λ̄1
λ̄1 = λ̄2
λ̄2 = λ̄
elseif (Δλ̄1 < Tol)    then
icase = 2
λ̄1 = λ̄3
λ̄3 = λ̄2
elseif (Δλ̄2 < Tol)    then
icase = 2
else
icase = 1
endif
```

- Computation of eigenprojections ($\mathbf{m}_i$). The algorithm is given as follows:

$$
\begin{aligned}
&\text{if (icase} \;=\; 1) \qquad \text{then}\\
\mathbf{m}_1 \;&=\; \frac{(\mathbf{b}^e)^2 - (\bar{\lambda}_2 + \bar{\lambda}_3)\mathbf{b}^e + \bar{\lambda}_2\bar{\lambda}_3\mathbf{i}}{(\bar{\lambda}_1 - \bar{\lambda}_3)(\bar{\lambda}_1 - \bar{\lambda}_2)}\\
\mathbf{m}_2 \;&=\; \frac{(\mathbf{b}^e)^2 - (\bar{\lambda}_1 + \bar{\lambda}_3)\mathbf{b}^e + \bar{\lambda}_1\bar{\lambda}_3\mathbf{i}}{(\bar{\lambda}_2 - \bar{\lambda}_3)(\bar{\lambda}_2 - \bar{\lambda}_1)}\\
\mathbf{m}_3 \;&=\; \frac{(\mathbf{b}^e)^2 - (\bar{\lambda}_1 + \bar{\lambda}_2)\mathbf{b}^e + \bar{\lambda}_1\bar{\lambda}_2\mathbf{i}}{(\bar{\lambda}_3 - \bar{\lambda}_1)(\bar{\lambda}_3 - \bar{\lambda}_2)}\\
&\text{elseif (icase} \;=\; 2) \qquad \text{then}\\
\mathbf{m}_1 \;&=\; \frac{\mathbf{b}^e - \bar{\lambda}_2\,\mathbf{i}}{\bar{\lambda}_1 - \bar{\lambda}_2}\\
\mathbf{m}_2 \;&=\; \frac{\mathbf{b}^e - \bar{\lambda}_1\,\mathbf{i}}{\bar{\lambda}_2 - \bar{\lambda}_1}\\
\mathbf{m}_3 \;&=\; \mathbf{m}_2\\
&\text{elseif (icase} \;=\; 3) \qquad \text{then}\\
\mathbf{m}_1 \;&=\; \mathbf{i}\\
\mathbf{m}_2 \;&=\; \mathbf{m}_1\\
\mathbf{m}_3 \;&=\; \mathbf{m}_1\\
&\text{endif}
\end{aligned}
$$

- Computation of $\mathbf{b}^e$ after accomplishment of the Return Map. The algorithm is given as follows:

$$
\begin{aligned}
&\text{if}\,(\text{icase} \;=\; 3) \qquad \text{then} \\
&\text{fakt}_{(1)} \;=\; \frac{1}{3} \\
&\text{fakt}_{(2)} \;=\; \frac{1}{3} \\
&\text{fakt}_{(3)} \;=\; \frac{1}{3} \\
&\text{elseif}\,(\text{icase} \;=\; 2) \qquad \text{then} \\
&\text{fakt}_{(1)} \;=\; 1 \\
&\text{fakt}_{(2)} \;=\; \frac{1}{2} \\
&\text{fakt}_{(3)} \;=\; \frac{1}{2} \\
&\text{else} \\
&\text{fakt}_{(1)} \;=\; 1 \\
&\text{fakt}_{(2)} \;=\; 1 \\
&\text{fakt}_{(3)} \;=\; 1 \\
&\text{endif} \\
&\text{the computation of } \mathbf{b}^e_{(n+1)} \text{ with prefactors} \\
&\mathbf{b}^e_{(n+1)} \;=\; \sum_{i=1}^{3} \bar{\lambda}_{i\,(n+1)}\,\text{fakt}_{(i)}\,\mathbf{m}_i
\end{aligned}
$$

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