Large Rotation Analysis of thin-walled shells

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Summary

The objective of this contribution is to develop an isoparametric 4-node shell element by means of an updated rotation procedure with 2/3 rotation parameters for a unified finite rotation analysis of smooth and compound shell structures.

Introduction

The purpose of this contribution is to develop a finite rotation shell element applicable in a unified form both to smooth and compound hyperelastic shells. The development starts from a six parametric shell kinematics including transverse strains [8]. This allows a direct enforcement of arbitrary constitutive laws into shell equations. In the present contribution NEO-HOOKEAN- hyperelastic materials are considered as example. The parametrization of the inextensible shell director entering in the kinematic assumption is achieved similar to [5] [7] by an updated rotation procedure with three independent rotation parameters defined with respect to a global reference frame. To consider shells with continous and discontinous director fields within a unified approach it is in the sequel distinguished between regular and irregular nodal points. In the regular nodal points located in the smooth shell the rotation parameters initially defined in the global reference frame are replaced through a transformation procedure by two tangential rotation variables [3]. This procedure removes the free energy modes involved in the final equation system and ensures a singularity-free simulation of shells involving geometry intersections. A further difficulty in the derivation of the updated rotation formulation is the expression obtained for the second variation of the director which causes in using three rotation parameters a non-symmetric equation. This difficulty is removed by an a priori symmetrization of the corresponding relation. Examples are given to demonstrate the prediction capability of the finite rotation element.

Basic Assumptions

The shell continuum is approximated by a linear expression in the thickness coordinate $\theta^3$. To consider transverse strains the first order term of this expression is multiplicatively decomposed into a stretch parameter $\lambda$ and a unit director. This leads to:
The base vectors of the deformed state can be evaluated, which can be used to construct arbitrary strain measures e.g. CAUCHY-GREEN-strain-tensor $E$ or ALMANSI-strain-tensor:

$$E = \frac{1}{2} (g_i \cdot g_j - G_{ij}) G^i \otimes G^j, \quad g_\alpha = a_\alpha + \theta^3 (\lambda d)_\alpha, \quad g_3 = \lambda d. \quad (2)$$

According to [10] we use a compressible NEO-HOOKE-model in terms of the invariants of the left CAUCHY-GREEN-tensor in the following form:

$$W = \frac{1}{2} \kappa \left( \ln \sqrt{\det b} \right)^2 + \frac{1}{2} \mu \left[ J^{-2/3} tr[b] - 3 \right]. \quad (3)$$

**Finite rotation**

The director $d$ is subjected to the inextensibility constraint $d \cdot d = 1$. This can be fulfilled by a suitable parametrization, i.e. by a transformation of $d$ into such rotational quantities ensuring an a priori satisfaction of the mentioned constraint. In this work we use for this purpose the updated rotation procedure. The essential idea is to determine the actual position of the director with respect to the foregoing one by means of a rotation vector $R$. Using the RODRIGUES rotation vector $\varphi \times$ the rotation tensor $R$ is given by:

$$R = I + \frac{\sin \varphi}{\varphi} \hat{\varphi} + \frac{1 - \cos \varphi}{\varphi^2} \hat{\varphi} \otimes \hat{\varphi} = \cos \varphi I + \frac{\sin \varphi}{\varphi} \hat{\varphi} + \frac{1 - \cos \varphi}{\varphi^2} \varphi \otimes \varphi, \quad \text{where} \quad \hat{\varphi} = \varphi \times. \quad (4)$$

During an iterative-incremental procedure the first variation of the director $\delta d$ as well as its linearization $\Delta \delta d$ are needed. Now, our purpose is to express these quantities in terms of a suitable rotation vector. For its definition we form the first variation of the orthogonality condition $RR^T = I$. This expression indicates that the tensor $\hat{W}$ is
a skew-symmetric tensor, so that we can express it in terms of its axial vector, which
is given by the incremental rotation vector $\delta\omega\times$.

$$\delta(R^TR) = \delta(I) = \delta RR^T + R\delta R = \delta \hat{W} + \delta \hat{W}^T = 0.$$  (5)

Accordingly we obtain:

$$d = RD, \quad \delta d = \delta RD = \delta RR^T RD = \delta RR^T d = \delta \hat{W} d = \delta \omega \times d.$$  (6)

and through linearization of the above expression:

$$\Delta \delta d = \Delta(\delta \hat{W} RD) = \delta \hat{W} \Delta \hat{W} d = \delta \omega \times (\Delta \omega \times d).$$  (7)

We distinguish between regular and irregular nodes, these lying on the smooth
shell and those on intersection curves. Since we employ a rotation vector with three
components within the entire shell defined with respect to a global reference frame,
we do not simplify in the case of regular nodes. The non-symmetry in the linearization
is omitted by symmetrizing this equation.

$$\Delta \delta d = \frac{1}{2}(\Delta \omega \times (\Delta \omega \times d) + \delta \omega \times (\Delta \omega \times d)).$$  (8)

After evaluation of an iteration step the new position of the director $d^{n+1}$ can
be determined in exact form with respect to the foregoing one $d^n$ through the relation
(4) with $\delta \omega$ instead of $\varphi$.

$$d^{n+1} = Rd^n.$$  (9)

**Finite Element Formulation**

For the numerical implementation the principle of virtual work is linearized
by the standard procedure and then transformed into a 4-node isoparametric shell ele-
ment by using bilinear interpolation polynomials. The integration over the thickness
coordinate $\theta^3$ is enforced numerically. To avoid locking-phenomena certain stabilization procedures are employed. To suppress shear locking the shear strains are approximated according to the assumed-strain-concept. To avoid membrane-, volume- and POISSON-locking we apply the EAS-concept [9] [6]. The basic concepts used for the implementation of the director can be summarized as follows. In the initial state the director $\mathbf{D}$ is identified with the unit normal vector of the exact mid-surface. Any nodal point is connected with a single director. In the actual state the director $\mathbf{d}$ is interpolated similar to the position vector by bilinear interpolation polynomials:

$$d = \sum_{a=1}^{4} N^a_d \mathbf{d}_a, \quad d_\alpha = \sum_{a=1}^{4} N^a_{d\alpha} \mathbf{d}_a. \quad (10)$$

The variation of the director and its linearization is expressed in terms of three rotation variables $\omega_i$ with respect to a global orthonormal reference frame. The constraints (6,7) are fulfilled at the nodal points (a denotes nodal point A):

$$\delta \mathbf{d} = \sum_{a=1}^{4} N^a \delta \mathbf{\omega}_a \times \mathbf{d}_a, \quad \Delta \delta \mathbf{d} = \sum_{a=1}^{4} \text{sym} [N^a \delta \mathbf{\omega}_a \times (N^a \Delta \mathbf{\omega}_a \times \mathbf{d}_a)]. \quad (11)$$

Once an iteration step is accomplished the new director is determined by updated rotation according to (9) at each node. Since no distinction is made between regular and irregular nodes so far we encounter zero-energy-modes at the regular nodal points due to the stiffnessfree rotation of the director about its own axis. Only at irregular nodes this is compensated, since several directors pointing in different directions are connected at one node. To omit the singularity at regular nodes we apply a procedure on the element level. Since two components in the plane perpendicular to the director are essential, we transform the rotation variables into a local frame with $\delta \omega_3$ in director direction. Next we set an artificial stiffness on the diagonal entry of the transformed stiffness matrix related to the component $\delta \omega_3$. After that we retransform the rotation variables back into the global frame.

**Examples**

**Large deflections of diamond-shaped frames** (Fig.1) This example is taken from [4], where an analytic solution and experimental data are given. A half of the square has been discretized with ten elements per length $L$ and two elements per height $H$. The displacement in $X^1$-direction has been blocked to avoid numerical instabilities.
The numerical results agree well with the analytic solution. This example shows, that the routines work properly.

\[ \eta^2 = \frac{qL^2}{EI} = 5 \cdot q \cdot \frac{2\Delta X_3}{L} \]

**Fig.1** Load parameter \( \eta^2 \) versus vertical displacement \( \Delta X_3 \)

**Sickle Shell problem** (Fig.2) This example is taken from [5]. It has been chosen to demonstrate the capability of the element to deal with large rotation. The analysis is performed up to the load level \( \lambda = 25 \) with equidistant steps of \( \Delta \lambda = 5 \).

**Geometry:**
- \( R=5.0 \)  \( L=10.0 \)  \( H = 1.0 \)  \( h = 0.01 \)

**Material data:**
- \( E=3.0 \cdot 10^7 \)  \( \nu=0.3 \)

**Load:**
- \( F=1 \cdot 10^{-3} \)

**Fig.2** Sickle shell problem at load factor \( \lambda = 25 \)

**Reference**


