Anisotropic elasto-plastic damage formulation for finite strains using a damage structural tensor and the effective stress concept

Olaf Kintzel and Günther Meschke

Institute for Structural Mechanics
Ruhr-University Bochum, Germany
Introduction

- Introduction
Introduction

- Introduction

- Introduction to basic concepts of continuum damage mechanics (CDM) typically used for small strains in form of equivalence principles
Introduction

- Introduction

- Introduction to basic concepts of continuum damage mechanics (CDM) typically used for small strains in form of equivalence principles

- Extension of these concepts to the finite strain case
Introduction

- Introduction

- Introduction to basic concepts of continuum damage mechanics (CDM) typically used for small strains in form of equivalence principles

- Extension of these concepts to the finite strain case

- Presentation of a consistent CDM model based on a straightforward extension of Lemaître's model to the anisotropic finite strain case using a damage tensor as a structural tensor
Introduction

- Introduction

- Introduction to basic concepts of continuum damage mechanics (CDM) typically used for small strains in form of equivalence principles

- Extension of these concepts to the finite strain case

- Presentation of a consistent CDM model based on a straightforward extension of LEMAITRE's model to the anisotropic finite strain case using a damage tensor as a structural tensor

- Conclusion
Continuum damage mechanics (CDM)

Description of a continuum damage process by means of a continuum damage parameter in the isotropic or anisotropic case using a scalar-valued or even-rank higher-order tensor.
Continuum damage mechanics (CDM)

Description of a continuum damage process by means of a continuum damage parameter in the isotropic or anisotropic case using a scalar-valued or even-rank higher-order tensor.

**small strain case** : Equivalence principles are typically used by introducing a fictive (undamaged) state of the material to motivate a transformation-relation for the stress tensor.
Continuum damage mechanics (CDM)

Description of a continuum damage process by means of a continuum damage parameter in the isotropic or anisotropic case using a scalar-valued or even-rank higher-order tensor.

**small strain case**: Equivalence principles are typically used by introducing a fictive (undamaged) state of the material to motivate a transformation-relation for the stress tensor.

- net stress concept (area reduction) [Betten 1986]
Continuum damage mechanics (CDM)

Description of a continuum damage process by means of a continuum damage parameter in the isotropic or anisotropic case using a scalar-valued or even-rank higher-order tensor.

**small strain case**: Equivalence principles are typically used by introducing a fictive (undamaged) state of the material to motivate a transformation-relation for the stress tensor.

- net stress concept (area reduction)  
[Betten 1986]

- stress equivalence principle  
[Simo & Ju 1987]
Continuum damage mechanics (CDM)

Description of a continuum damage process by means of a continuum damage parameter in the isotropic or anisotropic case using a scalar-valued or even-rank higher-order tensor.

**small strain case** : Equivalence principles are typically used by introducing a fictive (undamaged) state of the material to motivate a transformation-relation for the stress tensor.

- net stress concept (area reduction) [Betten 1986]
- stress equivalence principle [Simo & Ju 1987]
- energy equivalence principle [Cordebois & Sidoroff 1992]
Continuum damage mechanics (CDM)

Description of a continuum damage process by means of a continuum damage parameter in the isotropic or anisotropic case using a scalar-valued or even-rank higher-order tensor.

Small strain case: Equivalence principles are typically used by introducing a fictive (undamaged) state of the material to motivate a transformation-relation for the stress tensor.

- net stress concept (area reduction) [Betten 1986]
- stress equivalence principle [Simo & Ju 1987]
- energy equivalence principle [Cordebois & Sidoroff 1992]
- strain equivalence principle (effective stress concept) [Lemaitre 1992]
Continuum damage mechanics (CDM)

Description of a continuum damage process by means of a continuum damage parameter in the isotropic or anisotropic case using a scalar-valued or even-rank higher-order tensor.

**Small strain case**: Equivalence principles are typically used by introducing a fictive (undamaged) state of the material to motivate a transformation-relation for the stress tensor.

- net stress concept (area reduction) [Betten 1986]
- stress equivalence principle [Simo & Ju 1987]
- energy equivalence principle [Cordebois & Sidoroff 1992]
- strain equivalence principle (effective stress concept) [Lemaitre 1992]

**Finite strain case**: The fictive (undamaged) configuration introduced in the small strain case as necessary assumption can be interpreted as a single configuration which is reached within a suitable multiplicative decomposition of the deformation gradient.
Continuum damage mechanics (CDM)

Description of a continuum damage process by means of a continuum damage parameter in the isotropic or anisotropic case using a scalar-valued or even-rank higher-order tensor.

**small strain case**: Equivalence principles are typically used by introducing a fictive (undamaged) state of the material to motivate a transformation-relation for the stress tensor.

- net stress concept (area reduction) [Betten 1986]
- stress equivalence principle [Simo & Ju 1987]
- energy equivalence principle [Cordebois & Sidoroff 1992]
- strain equivalence principle (effective stress concept) [Lemaitre 1992]

**finite strain case**: The fictive (undamaged) configuration introduced in the small strain case as necessary assumption can be interpreted as a single configuration which is reached within a suitable multiplicative decomposition of the deformation gradient.

- Re-interpretation of the net stress concept for finite strains [Voyiadjis & Deliktas 2000]
Continuum damage mechanics (CDM)

Description of a continuum damage process by means of a continuum damage parameter in the isotropic or anisotropic case using a scalar-valued or even-rank higher-order tensor.

**small strain case**: Equivalence principles are typically used by introducing a fictive (undamaged) state of the material to motivate a transformation-relation for the stress tensor.

- net stress concept (area reduction) [Betten 1986]
- stress equivalence principle [Simo & Ju 1987]
- energy equivalence principle [Cordebois & Sidoroff 1992]
- strain equivalence principle (effective stress concept) [Lemaitre 1992]

**finite strain case**: The fictive (undamaged) configuration introduced in the small strain case as necessary assumption can be interpreted as a single configuration which is reached within a suitable multiplicative decomposition of the deformation gradient.

- Re-interpretation of the net stress concept for finite strains [Voyiadjis & Deliktas 2000]
- Re-interpretation of energy equivalence for finite strains [Menzel & Steinmann 2003]
Continuum damage mechanics (CDM)

Description of a continuum damage process by means of a continuum damage parameter in the isotropic or anisotropic case using a scalar-valued or even-rank higher-order tensor.

**small strain case**: Equivalence principles are typically used by introducing a fictive (undamaged) state of the material to motivate a transformation-relation for the stress tensor.

- net stress concept (area reduction) [Betten 1986]
- stress equivalence principle [Simo & Ju 1987]
- energy equivalence principle [Cordebois & Sidoroff 1992]
- strain equivalence principle (effective stress concept) [Lemaitre 1992]

**finite strain case**: The fictive (undamaged) configuration introduced in the small strain case as necessary assumption can be interpreted as a single configuration which is reached within a suitable multiplicative decomposition of the deformation gradient.

- Re-interpretation of the net stress concept for finite strains [Voyiadjis & Deliktas 2000]
- Re-interpretation of energy equivalence for finite strains [Menzel & Steinmann 2003]
- Re-interpretation of strain equivalence for finite strains [present model]
Net stress concept (area reduction)

real (damaged) state       fictive (pseudo-undamaged) state

- $e_2 \, dA_2$  
- $e_1 \, dA_1$  
- $e_3 \, dA_3$

n dA

$e_1$ 
$e_2$ 
$e_3$

$\tilde{n} \, d\tilde{A}$

$\tilde{e}_2 \, d\tilde{A}_2$  
$\tilde{e}_1 \, d\tilde{A}_1$  
$\tilde{e}_3 \, d\tilde{A}_3$
Net stress concept (area reduction)

\[
\begin{align*}
(1 - D_1) dA_1 &= d\tilde{A}_1 \\
(1 - D_2) dA_2 &= d\tilde{A}_2 \\
(1 - D_3) dA_3 &= d\tilde{A}_3
\end{align*}
\]

\[
(I - D) \mathbf{n} dA = \mathbf{\tilde{n}} d\tilde{A}
\]
Net stress concept (area reduction)

\[
\begin{align*}
(1 - D_1) \, dA_1 &= d\tilde{A}_1 \\
(1 - D_2) \, dA_2 &= d\tilde{A}_2 \\
(1 - D_3) \, dA_3 &= d\tilde{A}_3
\end{align*}
\]

\[
(I - D) \, n \, dA = \tilde{n} \, d\tilde{A}
\]

\[t = \tilde{t}\]
Net stress concept (area reduction)

\[
\begin{align*}
(1 - D_1) \, dA_1 &= d\tilde{A}_1 \\
(1 - D_2) \, dA_2 &= d\tilde{A}_2 \\
(1 - D_3) \, dA_3 &= d\tilde{A}_3
\end{align*}
\]

\[
(I - D) \, n \, dA = \tilde{n} \, d\tilde{A}
\]

\[
t = \tilde{t}
\]

\[
\sigma \, n \, dA = \tilde{\sigma} \, \tilde{n} \, d\tilde{A}
\]
Net stress concept (area reduction)

\[
\begin{align*}
(1 - D_1) dA_1 &= d\tilde{A}_1 \\
(1 - D_2) dA_2 &= d\tilde{A}_2 \\
(1 - D_3) dA_3 &= d\tilde{A}_3
\end{align*}
\]
\[
(I - D) n dA = \tilde{n} d\tilde{A}
\]
\[
t = \tilde{t}
\]
\[
\sigma n dA = \tilde{\sigma} \tilde{n} d\tilde{A} = \tilde{\sigma} (I - D) n dA
\]
Net stress concept (area reduction)

\[
\begin{align*}
(1 - D_1) \, dA_1 &= d\tilde{A}_1 \\
(1 - D_2) \, dA_2 &= d\tilde{A}_2 \\
(1 - D_3) \, dA_3 &= d\tilde{A}_3
\end{align*}
\]

\[
(I - D) \, n \, dA = \tilde{n} \, d\tilde{A} \quad \text{fictive (pseudo-undamaged) state}
\]

\[
t = \tilde{t}
\]

\[
\sigma \, n \, dA = \tilde{\sigma} \, \tilde{n} \, d\tilde{A} = \tilde{\sigma} \, (I - D) \, n \, dA \Rightarrow \tilde{\sigma} = \sigma (I - D)^{-1}
\]
Equivalence principles in general

\[ C^d, \mathbf{D} \]

\[ C^0, \mathbf{D}=0 \]

real (damaged) state   fictive (pseudo-undamaged)

state

\[ \sigma, \varepsilon \]

\[ \hat{\sigma}, \hat{\varepsilon} \]
Equivalence principles in general

\[ \psi = \frac{1}{2} \epsilon : C^d : \epsilon \quad \tilde{\psi} = \frac{1}{2} \tilde{\epsilon} : C^0 : \tilde{\epsilon} \]
Equivalence principles in general

\[ \mathcal{C}^d, \mathcal{D} \]
\[ \sigma, \varepsilon \]
real (damaged) state

\[ \mathcal{C}^0, \mathcal{D}=0 \]
\[ \tilde{\sigma}, \tilde{\varepsilon} \]
fictive (pseudo-undamaged) state

Energy
\[ \psi = \frac{1}{2} \varepsilon : \mathcal{C}^d : \varepsilon \]
\[ \tilde{\psi} = \frac{1}{2} \tilde{\varepsilon} : \mathcal{C}^0 : \tilde{\varepsilon} \]

Complementary energy
\[ \bar{\psi} = \frac{1}{2} \sigma : \mathcal{D}^d : \sigma \]
\[ \tilde{\bar{\psi}} = \frac{1}{2} \tilde{\sigma} : \mathcal{D}^0 : \tilde{\sigma} \]
Equivalence principles in general

\[ \psi = \frac{1}{2} \varepsilon : C^d : \varepsilon \quad \tilde{\psi} = \frac{1}{2} \tilde{\varepsilon} : C^0 : \tilde{\varepsilon} \]

\[ \bar{\psi} = \frac{1}{2} \sigma : D^d : \sigma \quad \tilde{\bar{\psi}} = \frac{1}{2} \tilde{\sigma} : D^0 : \tilde{\sigma} \]

\[ \sigma = \frac{\partial \psi}{\partial \varepsilon} = C^d : \varepsilon \quad \tilde{\sigma} = \frac{\partial \tilde{\psi}}{\partial \tilde{\varepsilon}} = C^0 : \tilde{\varepsilon} \]
Equivalence principles in general

\[ \psi = \frac{1}{2} \varepsilon : C^d : \varepsilon \quad \tilde{\psi} = \frac{1}{2} \tilde{\varepsilon} : C^0 : \tilde{\varepsilon} \]

Complementary energy

\[ \tilde{\psi} = \frac{1}{2} \sigma : D^d : \sigma \quad \tilde{\psi} = \frac{1}{2} \tilde{\sigma} : D^0 : \tilde{\sigma} \]

Stiffness relation

\[ \sigma = \frac{\partial \psi}{\partial \varepsilon} = C^d : \varepsilon \quad \tilde{\sigma} = \frac{\partial \tilde{\psi}}{\partial \tilde{\varepsilon}} = C^0 : \tilde{\varepsilon} \]

Flexibility relation

\[ \varepsilon = \frac{\partial \tilde{\psi}}{\partial \sigma} = D^d : \sigma \quad \tilde{\varepsilon} = \frac{\partial \tilde{\psi}}{\partial \tilde{\sigma}} = D^0 : \tilde{\sigma} \]
Stress equivalence principle

\[ \sigma(\epsilon, D) = \tilde{\sigma}(\tilde{\epsilon}, D = 0) \]

The stress associated with a damaged state under the applied strain \( \epsilon \) is equivalent to the stress associated with the undamaged state under the effective strain \( \tilde{\epsilon} \).
Stress equivalence principle

\[ \sigma(\epsilon, D) = \tilde{\sigma}(\tilde{\epsilon}, D = 0) \]

The stress associated with a damaged state under the applied strain \( \epsilon \) is equivalent to the stress associated with the undamaged state under the effective strain \( \tilde{\epsilon} \)

\[ \tilde{\epsilon} = D^0 \cdot C^d : \epsilon = M : \epsilon \]
Strain equivalence principle

\[ \epsilon(\sigma, D) = \tilde{\epsilon}(\tilde{\sigma}, D = 0) \]

The strain associated with a damaged state under the applied stress \( \sigma \) is equivalent to the strain associated with the undamaged state under the effective stress \( \tilde{\sigma} \)
Strain equivalence principle

\[ \epsilon(\sigma, D) = \tilde{\epsilon}(\tilde{\sigma}, D = 0) \]

The strain associated with a damaged state under the applied stress \( \sigma \) is equivalent to the strain associated with the undamaged state under the effective stress \( \tilde{\sigma} \)

\[ \tilde{\sigma} = C^0 : D^d : \sigma = \overline{M} : \sigma \]
Energy equivalence principle

\[ \psi(D) = \tilde{\psi}(D = 0) \]
Energy equivalence principle

\[ \psi(D) = \tilde{\psi}(D = 0) \]

if we assume \( \tilde{\epsilon} = \tilde{\mathbf{M}} : \epsilon \Rightarrow \)
Energy equivalence principle

\[ \psi(D) = \tilde{\psi}(D = 0) \]

if we assume \( \tilde{\varepsilon} = \tilde{M} : \varepsilon \Rightarrow \mathbb{C}^d = \tilde{M}^T : \mathbb{C}^0 : \tilde{M} \)
Energy equivalence principle

\[ \psi(D) = \tilde{\psi}(D = 0) \]

If we assume \( \tilde{\epsilon} = \tilde{\mathbf{M}} : \mathbf{\epsilon} \Rightarrow \]

\[ C^d = \tilde{\mathbf{M}}^T : C^0 : \tilde{\mathbf{M}} \]
\[ D^d = \tilde{\mathbf{M}}^{-1} : D^0 : \tilde{\mathbf{M}}^{-T} \]
Energy equivalence principle

\[ \psi(D) = \tilde{\psi}(D = 0) \]

if we assume \( \tilde{\epsilon} = \tilde{\mathbf{M}} : \epsilon \Rightarrow \)
\[
\begin{align*}
\mathbf{C}^d &= \tilde{\mathbf{M}}^T : \mathbf{C}^0 : \tilde{\mathbf{M}} \\
\mathbf{D}^d &= \tilde{\mathbf{M}}^{-1} : \mathbf{D}^0 : \tilde{\mathbf{M}}^{-T}
\end{align*}
\]
\[
\tilde{\sigma} = \mathbf{C}^0 : \tilde{\epsilon}
\]
Energy equivalence principle

\[
\psi(D) = \tilde{\psi}(D = 0)
\]

if we assume \( \tilde{\epsilon} = \tilde{\mathbf{M}} : \epsilon \Rightarrow C^d = \tilde{\mathbf{M}}^T : C^0 : \tilde{\mathbf{M}} \\
D^d = \tilde{\mathbf{M}}^{-1} : D^0 : \tilde{\mathbf{M}}^{-T} \\
\tilde{\sigma} = C^0 : \tilde{\epsilon} = \tilde{\mathbf{M}}^{-T} : C^d : \tilde{\mathbf{M}}^{-1} : \tilde{\epsilon}
\]
Energy equivalence principle

\[ \psi(D) = \tilde{\psi}(D = 0) \]

if we assume \( \tilde{\epsilon} = \tilde{\epsilon}_M : \epsilon \Rightarrow \)

\[ C^d = \tilde{\epsilon}_M^T : C^0 : \tilde{\epsilon}_M \]
\[ D^d = \tilde{\epsilon}_M^{-1} : D^0 : \tilde{\epsilon} \]
\[ \tilde{\sigma} = C^0 : \tilde{\epsilon} = \tilde{\epsilon}_M^{-T} : C^d : \tilde{\epsilon}_M^{-1} : \tilde{\epsilon} = \tilde{\epsilon}_M^{-T} : C^d : \epsilon \]
Energy equivalence principle

\[ \psi(D) = \tilde{\psi}(D = 0) \]

if we assume \( \tilde{\epsilon} = \tilde{\mathbf{M}} : \epsilon \Rightarrow \)

\[ \mathbb{C}^d = \tilde{\mathbf{M}}^T : \mathbb{C}^0 : \tilde{\mathbf{M}} \]
\[ \mathbb{D}^d = \tilde{\mathbf{M}}^{-1} : \mathbb{D}^0 : \tilde{\mathbf{M}}^{-T} \]

\[ \tilde{\sigma} = \mathbb{C}^0 : \tilde{\epsilon} = \tilde{\mathbf{M}}^{-T} : \mathbb{C}^d : \tilde{\mathbf{M}}^{-1} : \tilde{\epsilon} = \tilde{\mathbf{M}}^{-T} : \mathbb{C}^d : \epsilon = \tilde{\mathbf{M}}^{-T} : \sigma \]
Energy equivalence principle

\[ \psi(D) = \tilde{\psi}(D = 0) \]

if we assume \( \tilde{\epsilon} = \tilde{\mathbf{M}} : \epsilon \Rightarrow \)
\[ C^d = \tilde{\mathbf{M}}^T : C^0 : \tilde{\mathbf{M}} \]
\[ D^d = \tilde{\mathbf{M}}^{-1} : D^0 : \tilde{\mathbf{M}}^{-T} \]
\[ \tilde{\sigma} = C^0 : \tilde{\epsilon} = \tilde{\mathbf{M}}^{-T} : C^d : \tilde{\mathbf{M}}^{-1} : \tilde{\epsilon} = \tilde{\mathbf{M}}^{-T} : C^d : \epsilon = \tilde{\mathbf{M}}^{-T} : \sigma \]
\[ \tilde{\epsilon} = \tilde{\mathbf{M}} : \epsilon , \quad \tilde{\sigma} = \tilde{\mathbf{M}}^{-T} : \sigma \]
Equivalence principles in general

\[ C^d, D \quad \text{and} \quad C^0, D=0 \]

\[ \sigma, \varepsilon \quad \text{and} \quad \tilde{\sigma}, \tilde{\varepsilon} \]

real (damaged) state \quad fictive (pseudo-undamaged) state
Equivalence principles in general

\[ \mathbb{C}^d, D \]
\[ \sigma, \varepsilon \]
real (damaged) state

\[ \mathbb{C}^0, D=0 \]
\[ \tilde{\sigma}, \tilde{\varepsilon} \]
fictive (pseudo-undamaged) state

Net stress concept

\[ t = \tilde{t} \]
Equivalence principles in general

\[ C^d, D \]
\[ \sigma, \varepsilon \]
real (damaged) state

\[ C^0, D=0 \]
\[ \tilde{\sigma}, \tilde{\varepsilon} \]
fictive (pseudo-undamaged) state

Net stress concept
\[ t = \tilde{t} \]

Stress equivalence
\[ \tilde{\sigma} = \sigma \]
\[ \tilde{\varepsilon} = \mathbb{M} : \varepsilon \]
**Equivalence principles in general**

- Real (damaged) state: $C^d, D$
- Fictive (pseudo-undamaged) state: $C^0, D=0$

**Net stress concept**

$$t = \tilde{t}$$

**Stress equivalence**

$$\tilde{\sigma} = \sigma \quad \tilde{\epsilon} = M : \epsilon$$

**Strain equivalence**

$$\tilde{\sigma} = \bar{M} : \sigma \quad \tilde{\epsilon} = \epsilon$$
Equivalence principles in general

\[ C^d, D \quad \sigma, \varepsilon \]
real (damaged) state

\[ C^0, D=0 \quad \tilde{\sigma}, \tilde{\varepsilon} \]
fictive (pseudo-undamaged) state

Net stress concept
\[ t = \tilde{t} \]

Stress equivalence
\[ \tilde{\sigma} = \sigma \quad \tilde{\varepsilon} = \mathbb{M} : \varepsilon \]

Strain equivalence
\[ \tilde{\sigma} = \mathbb{M}^{-T} : \sigma \quad \tilde{\varepsilon} = \mathbb{M} : \varepsilon \]

Energy equivalence
Continuum damage mechanics (CDM)
Continuum damage mechanics (CDM)

- $D$ is usually assumed to be symmetrical
Continuum damage mechanics (CDM)

- $D$ is usually assumed to be symmetrical

- Symmetrization of Cauchy stress tensor $\tilde{\sigma} = \sigma (I - D)^{-1}$ (see net stress concept)
Continuum damage mechanics (CDM)

- $D$ is usually assumed to be symmetrical

- Symmetrization of Cauchy stress tensor ($\tilde{\sigma} = \sigma (I - D)^{-1}$ (see net stress concept))
  
  $\tilde{\sigma} = \frac{1}{2} \left( \sigma (I - D)^{-1} + (I - D)^{-1} \sigma \right)$  
  $\bar{\mathbf{m}} = \frac{1}{2} \left( (I - D)^{-1} \bigotimes I + I \bigotimes (I - D)^{-1} \right)$  
  [Murakami 1988]
Continuum damage mechanics (CDM)

• $D$ is usually assumed to be symmetrical

• Symmetrization of Cauchy stress tensor ($\tilde{\sigma} = \sigma (I - D)^{-1}$ (see net stress concept))

  • $\tilde{\sigma} = \frac{1}{2} \left( \sigma (I - D)^{-1} + (I - D)^{-1} \sigma \right)$
  
  • $\tilde{\sigma} = (I - D)^{-1/2} \sigma (I - D)^{-1/2}$

  \[
  \bar{M} = \frac{1}{2} \left( (I - D)^{-1} \boxtimes I + I \boxtimes (I - D)^{-1} \right) \\
  \text{[Murakami 1988]}
  \]

  \[
  \bar{M} = \left( (I - D)^{-1/2} \boxtimes (I - D)^{-1/2} \right) \\
  \text{[Cordebois & Sidoroff 1992]}
  \]
**Continuum damage mechanics (CDM)**

- **D** is usually assumed to be symmetrical

- Symmetrization of Cauchy stress tensor \( \tilde{\sigma} = \sigma (I - D)^{-1} \) (see net stress concept)

  - \( \tilde{\sigma} = \frac{1}{2} (\sigma (I - D)^{-1} + (I - D)^{-1} \sigma) \)
  
  \[
  \tilde{\sigma} = (I - D)^{-1/2} \sigma (I - D)^{-1/2}
  \]
  
  \[
  \tilde{\sigma} = (I - D)^{-1} \sigma (I - D)^{-1}
  \]

  - \( \tilde{\sigma} = \frac{1}{2} ((I - D)^{-1} \boldsymbol{\boxtimes} I + I \boldsymbol{\boxtimes} (I - D)^{-1}) \)
  
  \[
  \tilde{\sigma} = (I - D)^{-1/2} \boldsymbol{\boxtimes} (I - D)^{-1/2}
  \]
  
  \[
  \tilde{\sigma} = (I - D)^{-1} \boldsymbol{\boxtimes} (I - D)^{-1}
  \]

  - [Murakami 1988]
  
  - [Cordebois & Sidoroff 1992]
  
  - [Betten 1986]
Continuum damage mechanics (CDM)

- \( D \) is usually assumed to be symmetrical

- Symmetrization of Cauchy stress tensor (\( \tilde{\sigma} = \sigma (I - D)^{-1} \) (see net stress concept))

  - \( \tilde{\sigma} = \frac{1}{2} (\sigma (I - D)^{-1} + (I - D)^{-1} \sigma) \)
  - \( \tilde{\sigma} = (I - D)^{-1/2} \sigma (I - D)^{-1/2} \)
  - \( \tilde{\sigma} = (I - D)^{-1} \sigma (I - D)^{-1} \)

\[ \tilde{\sigma} = \frac{1}{2} (\sigma (I - D)^{-1} + (I - D)^{-1} \sigma) \quad \tilde{\sigma} = (I - D)^{-1/2} \sigma (I - D)^{-1/2} \quad \tilde{\sigma} = (I - D)^{-1} \sigma (I - D)^{-1} \]

\[ \overline{M} = \frac{1}{2} \left( (I - D)^{-1} \overset{\bigotimes}{I} + I \overset{\bigotimes}{(I - D)^{-1}} \right) \quad \overline{M} = \left( (I - D)^{-1/2} \overset{\bigotimes}{I} (I - D)^{-1/2} \right) \quad \overline{M} = \left( (I - D)^{-1} \overset{\bigotimes}{I} (I - D)^{-1} \right) \]

[Cordebois & Sidoroff 1992]
[Betten 1986]

\( \Rightarrow \) The whole approach seems to be a little bit arbitrary.
Finite strain plasticity model

$S_{,E,\hat{G}}$: 2nd Piola-Kirchhoff stress tensor

$G, E$: material metric tensor, Green-Lagrange strain tensor

$\hat{S}_{,\hat{E},\hat{G}}$: 2nd Piola-Kirchhoff stress tensor of the intermediate configuration

$\hat{G}, \hat{E}$: metric tensor, Green-Lagrange strain tensor of the intermediate configuration

$\tau$: Kirchhoff stress tensor

$g, e$: spatial metric tensor, Almansi strain tensor
Finite strain plasticity model 1

\[ F = \sqrt{\text{dev}(\hat{\mathbf{C}}\hat{\mathbf{S}} - \hat{\mathbf{Y}})\text{dev}(\hat{\mathbf{C}}\hat{\mathbf{S}} - \hat{\mathbf{Y}})} : \mathbf{I} - \sqrt{\frac{2}{3}} q_1(\alpha) \leq 0 \]

\( \hat{\mathbf{C}}\hat{\mathbf{S}} \): Mandel-type stress tensor
\( \hat{\mathbf{C}} \): elastic right Cauchy-Green tensor
\( \hat{\mathbf{Y}} \): back stress tensor to consider kinematic hardening
\( q_1 \): isotropic hardening term, radius of the yield surface

[Itskov 2004]
Finite strain plasticity model 2

\[ F = \sqrt{\text{dev}(\hat{G}(\hat{\Sigma} - \hat{\kappa})) \text{dev}(\hat{G}(\hat{\Sigma} - \hat{\kappa}))} : \hat{I} - \sqrt{\frac{2}{3}} \, q_1(\alpha) \leq 0 \]  

[Dettmer & Reese 2004]

\(\hat{\Sigma}\) : \(\hat{\Sigma} = \hat{G}^{-1}\hat{C}\hat{S}\) symmetrical Mandel-type stress tensor considering isotropic hyperelasticity

\(\hat{G}\) : metric tensor of the intermediate state

\(\hat{\kappa}\) : symmetrical back stress tensor to consider kinematic hardening consistent with model 1

\(q_1\) : isotropic hardening term, radius of the yield surface
Finite strain plasticity model 3

\[ F = \sqrt{\text{dev}(g(\tau - \rho)) \text{dev}(g(\tau - \rho))} : i - \sqrt{\frac{2}{3}} q_1(\alpha) \leq 0 \]  

[Kintzel & Meschke 2005]

- \(\tau\): Kirchhoff stress tensor
- \(g\): spatial metric tensor
- \(\rho\): back stress tensor to consider kinematic hardening consistent with model 1 and model 2
- \(q_1\): isotropic hardening term, radius of the yield surface
Finite strain continuum damage mechanics

\[ \begin{align*}
\bar{S} & : \text{2nd Piola-Kirchhoff stress tensor of the fictive (intermediate) configuration} \\
\bar{G}, \bar{E} & : \text{metric tensor, Green-Lagrange strain tensor of the fictive (intermediate) configuration} \\
\bar{F}^d & : \text{damage deformation gradient} \\
\bar{F}^p & : \text{fictive plastic deformation gradient}
\end{align*} \]
Finite strain continuum damage mechanics

\[ \tilde{\tau}, \tilde{\varepsilon}, \tilde{\gamma} \]: Kirchhoff stress tensor of the fictive (spatial) configuration
\[ \tilde{\varepsilon}, \tilde{\varepsilon}, \tilde{\gamma} \]: metric tensor, Almansi strain tensor of the fictive (spatial) configuration
\[ \tilde{F}^d \]: damage deformation gradient
\[ \tilde{F}^e \]: fictive elastic deformation gradient
Re-interpretation of the net stress concept for finite strains [Voyiadjis & Deliktas 2000]

Possible extension to finite strains using Nanson’s formula: \( \mathbf{n} \, dA = \det(\tilde{\mathbf{F}}^d) \, \tilde{\mathbf{F}}^{d-T} \, \tilde{\mathbf{n}} \, d\tilde{A} \)
Re-interpretation of the net stress concept for finite strains [Voyiadjis & Deliktas 2000]

Possible extension to finite strains using Nanson's formula:

\[ (I - D) \, n \, dA = \tilde{n} \, d\tilde{A} \Rightarrow \tilde{F}^d = \frac{(I - D)}{\sqrt{\det(I - D)}} \]
Re-interpretation of the net stress concept for finite strains [Voyiadjis & Deliktas 2000]

Possible extension to finite strains using Nanson’s formula: \[ n\, dA = \det(\tilde{F}^d) \tilde{F}^{d-T} \tilde{n}\, d\tilde{A} \]

\[ (I - D)\, n\, dA = \tilde{n}\, d\tilde{A} \quad \Rightarrow \quad \tilde{F}^d = \frac{(I - D)}{\sqrt{\det(I - D)}} \]

\[ \tilde{\tau} = \tilde{F}^{d<}(\tau) = (I - D)^{-1}\tau(I - D)^{-1} \] \[ \text{effective Kirchhoff stress tensor} \]
Re-interpretation of the net stress concept for finite strains [Voyiadjis & Deliktas 2000]

Possible extension to finite strains using Nanson’s formula: \( n \, dA = \det(\tilde{F}^{d}) \tilde{F}^{d-T} \tilde{n} \, d\tilde{A} \)

\[
(I - D) \, n \, dA = \tilde{n} \, d\tilde{A} \quad \Rightarrow \quad \tilde{F}^{d} = \frac{(I - D)}{\sqrt{\det(I - D)}}
\]

\[
\tilde{\tau} = \tilde{F}^{d \cdot \tau} (\tau) = (I - D)^{-1} \tau (I - D)^{-1} \det(I - D)
\] effective Kirchhoff stress tensor

\[
\tilde{\sigma} = \sqrt{\det(I - D)} (I - D)^{-1} \sigma (I - D)^{-1}
\] effective Cauchy stress tensor
Re-interpretation of energy equivalence for finite strains [Menzel & Steinmann 2003]

\[ \bar{\psi}(\bar{E}, D = 0) = \hat{\psi}(\hat{E}, D) \]
Re-interpretation of energy equivalence for finite strains [Menzel & Steinmann 2003]

\[ \bar{\psi}(\bar{E}, D = 0) = \hat{\psi}(\hat{E}, D) \Rightarrow \bar{\psi}(\bar{E}, \bar{G}^{-1}) = \hat{\psi}(\bar{F}_d^d(\bar{E}), \bar{F}_d^d(\bar{G}^{-1})) = \hat{\psi}(\hat{E}, \hat{A}) \]

Construction of an isotropic strain energy function in the fictive configuration in terms of \( \bar{E} \) with respect to the fictive metric tensor \( \bar{G} \).
Re-interpretation of energy equivalence for finite strains [Menzel & Steinmann 2003]

Construction of an isotropic strain energy function in the fictive configuration in terms of $\bar{E}$ with respect to the fictive metric tensor $\bar{G}$. By assuming the damage deformation gradient in the form:

$$\bar{F}^d = \alpha_0 \bar{G}^{-1} + \sum_{j=1}^{2} \alpha_j \hat{A}_j$$
Re-interpretation of energy equivalence for finite strains [Menzel & Steinmann 2003]

Construction of an isotropic strain energy function in the fictive configuration in terms of $\overline{E}$ with respect to the fictive metric tensor $\overline{G}$. By assuming the damage deformation gradient in the form:

$$\overline{F}^d = \alpha_0 \hat{G}^{-1} + \sum_{j=1}^{2} \alpha_j \hat{A}_j$$

We obtain an anisotropic (damage) metric tensor in the intermediate configuration:

$$\hat{A} = \overline{F}^d_{\triangleright} (\overline{G}^{-1}) = \beta_0 \hat{G}^{-1} + \beta_1 \hat{A}_1 + \beta_2 \hat{A}_2 + 2 \beta_3 (\hat{A}_1 \hat{G} \hat{A}_2)_{\text{sym}}$$
Re-interpretation of energy equivalence for finite strains [Menzel & Steinmann 2003]

The same argument can be applied for the inelastic potential:

\[ F(\tilde{\tau}, \tilde{g}) = F(\tilde{\tau}, \tilde{g}) = F(\tau, p) \]

Construction of an isotropic inelastic potential in terms of \( \tilde{\tau} \) with respect to the fictive spatial metric tensor \( \tilde{g} \).
Re-interpretation of energy equivalence for finite strains [Menzel & Steinmann 2003]

The same argument can be applied for the inelastic potential:

\[ F(\tilde{\tau}, \tilde{g}) = F(\tilde{F}_d^d(\tilde{\tau}), \tilde{F}_d^d(\tilde{g})) = F(\tau, p) \]

Construction of an isotropic inelastic potential in terms of \( \tilde{\tau} \) with respect to the fictive spatial metric tensor \( \tilde{g} \). The damage deformation gradient \( \tilde{F}_d^d \) is defined by:

\[ \tilde{F}_d^d = F^e F_d^d F^e -1 \]
Re-interpretation of energy equivalence for finite strains [Menzel & Steinmann 2003]

The same argument can be applied for the inelastic potential:

\[ F(\bar{\tau}, \tilde{g}) = F(\tilde{\bar{F}}^d(\bar{\tau}), \tilde{\bar{F}}^d(\tilde{g})) = F(\tau, p) \]

Construction of an isotropic inelastic potential in terms of \( \bar{\tau} \) with respect to the fictive spatial metric tensor \( \tilde{g} \). The damage deformation gradient \( \tilde{\bar{F}}^d \) is defined by:

\[ \tilde{\bar{F}}^d = F^e \bar{F}^d F^e F^{-1} = F^e \tilde{\bar{F}}^d F^e F^{-1} \]

The additional, but not necessary, assumption \( F^e = \tilde{F}^e \) is made.
Re-interpretation of strain equivalence for finite strains [present model]

The present model can be considered as a direct extension of LEMAITRE’s model.
Re-interpretation of strain equivalence for finite strains [present model]

The present model can be considered as a direct extension of Lemaître’s model.

- Construction of the complementary energy function in terms of the 2nd Piola-Kirchhoff stress tensor $\hat{S}$ and a damage structural tensor $(\hat{G} - \hat{D})^{-1}$ as a direct extension of the isotropic model of Lemaître:

$$\psi = (1 - d) \varepsilon : C^0 : \varepsilon = (1 - d)^{-1} \sigma : D^0 : \sigma$$
The present model can be considered as a direct extension of Lemaître’s model.

- Construction of the complementary energy function in terms of the 2nd Piola-Kirchhoff stress tensor $\hat{S}$ and a damage structural tensor $(\hat{G} - \hat{D})^{-1}$ as a direct extension of the isotropic model of Lemaître:
  \[
  \psi = (1 - d) \epsilon : C^0 : \epsilon = (1 - d)^{-1} \sigma : D^0 : \sigma
  \]

- By assuming strain equivalence $\tilde{C} = \hat{C}$ the effective stress tensor $\tilde{S}$ is automatically symmetrical
Re-interpretation of strain equivalence for finite strains [present model]

The present model can be considered as a direct extension of Lemaître’s model.

- Construction of the complementary energy function in terms of the 2nd Piola-Kirchhoff stress tensor $\hat{\mathbf{S}}$ and a damage structural tensor $(\hat{\mathbf{G}} - \hat{\mathbf{D}})^{-1}$ as a direct extension of the isotropic model of Lemaître:

$$\psi = (1 - d) \varepsilon : C^0 : \varepsilon = (1 - d)^{-1} \sigma : D^0 : \sigma$$

- By assuming strain equivalence $\tilde{\mathbf{C}} = \hat{\mathbf{C}}$ the effective stress tensor $\tilde{\mathbf{S}}$ is automatically symmetrical.

- Consideration of the effective stress tensor $\tilde{\mathbf{S}}$ instead of $\hat{\mathbf{S}}$ as argument tensor in the inelastic potential.
Re-interpretation of strain equivalence for finite strains [present model]

Using the representation theorem for isotropic tensor functions we make use of the following joint invariants of $\hat{S}$ and the damage structural tensor $(\hat{G} - \hat{D})^{-1}$:

$$\text{tr}(\hat{S}) = \hat{G} : \hat{S}, \quad \text{tr}(\hat{S}^2) = \hat{G} : (\hat{S}\hat{G}\hat{S}), \quad \text{tr}((\hat{G} - \hat{D})^{-1}) = \hat{G} : (\hat{G} - \hat{D})^{-1}$$

$$\text{tr}(\hat{S}(\hat{G} - \hat{D})^{-1}) = \hat{S}\hat{G} : \hat{G}(\hat{G} - \hat{D})^{-1}, \quad \text{tr}(\hat{S}^2(\hat{G} - \hat{D})^{-1}) = \hat{S}\hat{G}\hat{S}\hat{G} : \hat{G}(\hat{G} - \hat{D})^{-1}$$
Re-interpretation of strain equivalence for finite strains [present model]

Using the representation theorem for isotropic tensor functions we make use of the following joint invariants of $\hat{S}$ and the damage structural tensor $(\hat{G} - \hat{D})^{-1}$:

$$
\text{tr}(\hat{S}) = \hat{G} : \hat{S}, \quad \text{tr}(\hat{S}^2) = \hat{G} : (\hat{S}\hat{G}\hat{S}), \quad \text{tr}((\hat{G} - \hat{D})^{-1}) = \hat{G} : (\hat{G} - \hat{D})^{-1}
$$

$$
\text{tr}(\hat{S}(\hat{G} - \hat{D})^{-1}) = \hat{S}\hat{G} : \hat{G}(\hat{G} - \hat{D})^{-1}, \quad \text{tr}(\hat{S}^2(\hat{G} - \hat{D})^{-1}) = \hat{S}\hat{G}\hat{S}\hat{G} : \hat{G}(\hat{G} - \hat{D})^{-1}
$$

Constructing the isotropic St. Venant-Kirchhoff stiffness (flexibility) tensors:

$$
\mathbb{C}^0(\hat{G}, \hat{D} = 0) = \lambda \hat{G}^{-1} \otimes \hat{G}^{-1} + \mu \left( \hat{G}^{-1} \times \hat{G}^{-1} + \hat{G}^{-1} \boxtimes \hat{G}^{-1} \right)
$$

$$
\mathbb{D}^0(\hat{G}, \hat{D} = 0) = \lambda \hat{G} \otimes \hat{G} + \mu \left( \hat{G} \times \hat{G} + \hat{G} \boxtimes \hat{G} \right)
$$
Re-interpretation of strain equivalence for finite strains \[\text{[present model]}\]

Using the representation theorem for isotropic tensor functions we make use of the following joint invariants of \( \hat{S} \) and the damage structural tensor \((\hat{G} - \hat{D})^{-1}\):

\[
\begin{align*}
\text{tr}(\hat{S}) &= \hat{G} : \hat{S}, \\
\text{tr}(\hat{S}^2) &= \hat{G} : (\hat{S} \hat{G} \hat{S}), \\
\text{tr}(\hat{S}(\hat{G} - \hat{D})^{-1}) &= \hat{S} \hat{G} : (\hat{G} - \hat{D})^{-1} \\
\text{tr}(\hat{S}(\hat{G} - \hat{D})^{-1}) &= \hat{S} \hat{G} : (\hat{G} - \hat{D})^{-1} \\
\text{tr}(\hat{S}^2(\hat{G} - \hat{D})^{-1}) &= \hat{S} \hat{G} : (\hat{G} - \hat{D})^{-1} \\
\end{align*}
\]

Constructing the isotropic St. Venant-Kirchhoff stiffness (flexibility) tensors:

\[
\begin{align*}
\mathbb{C}^0(\hat{G}, \hat{D} = 0) &= \lambda \hat{G}^{-1} \otimes \hat{G}^{-1} + \mu \left( \hat{G}^{-1} \times \hat{G}^{-1} + \hat{G}^{-1} \boxtimes \hat{G}^{-1} \right) \\
\mathbb{D}^0(\hat{G}, \hat{D} = 0) &= \tilde{\lambda} \hat{G} \otimes \hat{G} + \tilde{\mu} \left( \hat{G} \times \hat{G} + \hat{G} \boxtimes \hat{G} \right)
\end{align*}
\]

By considering the relation:

\[
\mathbb{D}^0 : \mathbb{C}^0 = \frac{1}{2} (\hat{I} \times \hat{I} + \hat{I} \boxtimes \hat{I})
\]

the Lame’ constants obey the following relationship:

\[
\begin{align*}
\tilde{\lambda} &= \frac{-\lambda}{2 \mu (3 \lambda + 2 \mu)}, \\
\tilde{\mu} &= \frac{1}{4 \mu}
\end{align*}
\]
Re-interpretation of strain equivalence for finite strains \[\text{[present model]}\]

Construction of the anisotropic St. Venant-Kirchhoff stress energy function quadratically in \(\hat{\mathbf{S}}\) and linear in \((\hat{\mathbf{G}} - \hat{\mathbf{D}})^{-1}\):

\[
\hat{\psi}^e = \left( \eta_1 + \eta_2 \text{tr}(\hat{\mathbf{G}} - \hat{\mathbf{D}})^{-1} \right) \left( \text{tr}(\hat{\mathbf{S}}) \right)^2 + \left( \eta_3 + \eta_4 \text{tr}(\hat{\mathbf{G}} - \hat{\mathbf{D}})^{-1} \right) \text{tr}(\hat{\mathbf{S}})^2 \\
+ \eta_5 \text{tr}(\hat{\mathbf{S}}(\hat{\mathbf{G}} - \hat{\mathbf{D}})^{-1}) \text{tr}(\hat{\mathbf{S}}) + \eta_6 \text{tr}(\hat{\mathbf{S}}^2(\hat{\mathbf{G}} - \hat{\mathbf{D}})^{-1})
\]
Re-interpretation of strain equivalence for finite strains [present model]

Construction of the anisotropic St. Venant-Kirchhoff stress energy function quadratically in $\hat{\mathbf{S}}$ and linear in $(\hat{\mathbf{G}} - \hat{\mathbf{D}})^{-1}$:

$$
\hat{\psi}^e = \left( \eta_1 + \eta_2 \text{tr}((\hat{\mathbf{G}} - \hat{\mathbf{D}})^{-1}) \right) (\text{tr}(\hat{\mathbf{S}}))^2 + \left( \eta_3 + \eta_4 \text{tr}((\hat{\mathbf{G}} - \hat{\mathbf{D}})^{-1}) \right) \text{tr}(\hat{\mathbf{S}})^2 \\
+ \eta_5 \text{tr}(\hat{\mathbf{S}}(\hat{\mathbf{G}} - \hat{\mathbf{D}})^{-1})\text{tr}(\hat{\mathbf{S}}) + \eta_6 \text{tr}(\hat{\mathbf{S}}^2(\hat{\mathbf{G}} - \hat{\mathbf{D}})^{-1})
$$

By means of tensor differentiation rules we obtain the first-order derivative:

$$
\frac{\partial \hat{\psi}^e}{\partial \hat{\mathbf{S}}} = 2 \left( \eta_1 + \eta_2 (\hat{\mathbf{G}} - \hat{\mathbf{D}})^{-1} \right) \text{tr}(\hat{\mathbf{S}}) \hat{\mathbf{G}} + 2 \left( \eta_3 + \eta_4 \text{tr}(\hat{\mathbf{G}} - \hat{\mathbf{D}})^{-1} \right) \hat{\mathbf{G}} \hat{\mathbf{S}} \hat{\mathbf{G}} \\
+ \eta_5 \left( \text{tr}(\hat{\mathbf{S}}(\hat{\mathbf{G}} - \hat{\mathbf{D}})^{-1})\hat{\mathbf{G}} + \text{tr}(\hat{\mathbf{S}}) \hat{\mathbf{G}}(\hat{\mathbf{G}} - \hat{\mathbf{D}})^{-1}\hat{\mathbf{G}} \right) \\
+ \eta_6 \left( \hat{\mathbf{G}}(\hat{\mathbf{G}} - \hat{\mathbf{D}})^{-1}\hat{\mathbf{G}} \hat{\mathbf{S}} \hat{\mathbf{G}} + \hat{\mathbf{G}} \hat{\mathbf{S}} \hat{\mathbf{G}}(\hat{\mathbf{G}} - \hat{\mathbf{D}})^{-1}\hat{\mathbf{G}} \right)
$$
Re-interpretation of strain equivalence for finite strains [present model]

Construction of the anisotropic St. Venant-Kirchhoff stress energy function quadratically in \( \hat{S} \) and linear in \( (\hat{G} - \hat{D})^{-1} \):

\[
\hat{\psi}^e = \left( \eta_1 + \eta_2 \text{tr}((\hat{G} - \hat{D})^{-1}) \right) (\text{tr}(\hat{S}))^2 + \left( \eta_3 + \eta_4 \text{tr}((\hat{G} - \hat{D})^{-1}) \right) \text{tr}(\hat{S})^2 \\
+ \eta_5 \text{tr}(\hat{S}(\hat{G} - \hat{D})^{-1})\text{tr}(\hat{S}) + \eta_6 \text{tr}(\hat{S}^2(\hat{G} - \hat{D})^{-1})
\]
Re-interpretation of strain equivalence for finite strains \textit{[present model]}

Construction of the anisotropic \textit{St. Venant-Kirchhoff} stress energy function quadratically in $\hat{\mathbf{S}}$ and linear in $(\hat{\mathbf{G}} - \hat{\mathbf{D}})^{-1}$:

$$
\hat{\psi}^e = \left( \eta_1 + \eta_2 \text{tr}((\hat{\mathbf{G}} - \hat{\mathbf{D}})^{-1}) \right) (\text{tr}(\hat{\mathbf{S}}))^2 + \left( \eta_3 + \eta_4 \text{tr}((\hat{\mathbf{G}} - \hat{\mathbf{D}})^{-1}) \right) \text{tr}(\hat{\mathbf{S}})^2
$$

$$
+ \eta_5 \text{tr}(\hat{\mathbf{S}}(\hat{\mathbf{G}} - \hat{\mathbf{D}})^{-1}) \text{tr}(\hat{\mathbf{S}}) + \eta_6 \text{tr}(\hat{\mathbf{S}}^2(\hat{\mathbf{G}} - \hat{\mathbf{D}})^{-1})
$$

By means of tensor differentiaton rules we obtain the second-order derivative:

$$
\mathbb{D}^d = 2 \left( \eta_1 + \eta_2 \text{tr}((\hat{\mathbf{G}} - \hat{\mathbf{D}})^{-1}) \right) \hat{\mathbf{G}} \otimes \hat{\mathbf{G}} + \left( \eta_3 + \eta_4 \text{tr}((\hat{\mathbf{G}} - \hat{\mathbf{D}})^{-1}) \right) \left( \hat{\mathbf{G}} \otimes \hat{\mathbf{G}} + \hat{\mathbf{G}} \times \hat{\mathbf{G}} \right)
$$

$$
+ \eta_5 \left( \hat{\mathbf{G}} \otimes \hat{\mathbf{G}}(\hat{\mathbf{G}} - \hat{\mathbf{D}})^{-1} \hat{\mathbf{G}} + \hat{\mathbf{G}}(\hat{\mathbf{G}} - \hat{\mathbf{D}})^{-1} \hat{\mathbf{G}} \otimes \hat{\mathbf{G}} \right)
$$

$$
+ \eta_6 \frac{1}{2} \left( \hat{\mathbf{G}}(\hat{\mathbf{G}} - \hat{\mathbf{D}})^{-1} \hat{\mathbf{G}} \otimes \hat{\mathbf{G}} + \hat{\mathbf{G}}(\hat{\mathbf{G}} - \hat{\mathbf{D}})^{-1} \hat{\mathbf{G}} \times \hat{\mathbf{G}}
$$

$$
+ \hat{\mathbf{G}} \otimes \hat{\mathbf{G}}(\hat{\mathbf{G}} - \hat{\mathbf{D}})^{-1} \hat{\mathbf{G}} + \hat{\mathbf{G}} \times \hat{\mathbf{G}}(\hat{\mathbf{G}} - \hat{\mathbf{D}})^{-1} \hat{\mathbf{G}} \right)
$$
Re-interpretation of strain equivalence for finite strains [present model]

The anisotropic model should be consistent with LEMAITRE’s model for isotropic damage $\mathbf{D} = d \mathbf{G}$:
Re-interpretation of strain equivalence for finite strains [present model]

The anisotropic model should be consistent with Lemaître’s model for isotropic damage $\hat{D} = d \hat{G}$:

$$
2 \eta_1 + \frac{6 \eta_2}{(1 - d)} + \frac{2 \eta_5}{(1 - d)} = \frac{\bar{\lambda}}{(1 - d)}, \quad \eta_3 + \frac{3 \eta_4}{(1 - d)} + \frac{\eta_6}{(1 - d)} = \frac{\bar{\mu}}{(1 - d)}
$$
Re-interpretation of strain equivalence for finite strains [present model]

The anisotropic model should be consistent with Lemaître’s model for isotropic damage $\hat{D} = d \hat{G}$:

$$
2 \eta_1 + \frac{6 \eta_2}{(1 - d)} + \frac{2 \eta_5}{(1 - d)} = \frac{\bar{\lambda}}{(1 - d)}, \quad \eta_3 + \frac{3 \eta_4}{(1 - d)} + \frac{\eta_6}{(1 - d)} = \frac{\bar{\mu}}{(1 - d)}
$$

which suggests that $\eta_1 = \eta_3 = 0$. 
Re-interpretation of strain equivalence for finite strains [present model]

The anisotropic model should be consistent with LEMAÎTRE’s model for isotropic damage \( \hat{D} = d \hat{G} \):

\[
2 \eta_1 + \frac{6 \eta_2}{(1 - d)} + \frac{2 \eta_5}{(1 - d)} = \frac{\lambda}{(1 - d)}, \quad \eta_3 + \frac{3 \eta_4}{(1 - d)} + \frac{\eta_6}{(1 - d)} = \frac{\mu}{(1 - d)}
\]

which suggests that \( \eta_1 = \eta_3 = 0 \).

A further important restriction is convexity of \( \hat{\psi}^e \). We consider the flexibility matrix in VOIGT-notation by assuming that \( \hat{D} \) is decomposed in principal axes:

\[
\begin{pmatrix}
  \hat{D}_{111}^d & \hat{D}_{112}^d & \hat{D}_{113}^d & 0 & 0 & 0 \\
  \hat{D}_{221}^d & \hat{D}_{222}^d & \hat{D}_{223}^d & 0 & 0 & 0 \\
  \hat{D}_{331}^d & \hat{D}_{332}^d & \hat{D}_{333}^d & 0 & 0 & 0 \\
  0 & 0 & 0 & 2 \hat{D}_{1212}^d & 0 & 0 \\
  0 & 0 & 0 & 0 & 2 \hat{D}_{1313}^d & 0 \\
  0 & 0 & 0 & 0 & 0 & 2 \hat{D}_{2323}^d 
\end{pmatrix}
\]
Re-interpretation of strain equivalence for finite strains [present model]

\[
\begin{align*}
\det & \left( \begin{array}{ccc}
\frac{2}{(1-d_1)} (\eta_2 + \eta_4 + \eta_5 + \eta_6) & \frac{2}{(1-d_1)} (\eta_2 + \eta_5) & \frac{2}{(1-d_1)} (\eta_2 + \eta_5) \\
+ \frac{2}{(1-d_2)} (\eta_2 + \eta_4) & + \frac{2}{(1-d_2)} (\eta_2 + \eta_5) & + \frac{2}{(1-d_2)} (\eta_2) \\
+ \frac{2}{(1-d_3)} (\eta_2) & + \frac{2}{(1-d_3)} (\eta_2 + \eta_5) & + \frac{2}{(1-d_3)} (\eta_2 + \eta_5)
\end{array} \right) \\
\geq 0
\end{align*}
\]

\[
\begin{align*}
\frac{1}{(1-d_1)} (2\eta_4 + \eta_6) & + \frac{1}{(1-d_2)} (2\eta_4 + \eta_6) + \frac{1}{(1-d_3)} 2\eta_4 \geq 0 \\
\frac{1}{(1-d_1)} (2\eta_4 + \eta_6) & + \frac{1}{(1-d_2)} 2\eta_4 + \frac{1}{(1-d_3)} (2\eta_4 + \eta_6) \geq 0 \\
\frac{1}{(1-d_1)} 2\eta_4 & + \frac{1}{(1-d_2)} (2\eta_4 + \eta_6) + \frac{1}{(1-d_3)} (2\eta_4 + \eta_6) \geq 0
\end{align*}
\]

should be satisfied!
Re-interpretation of strain equivalence for finite strains [present model]

The strain associated with a damaged state under the applied stress $\hat{S}$ is equivalent to the strain associated with the undamaged state under the effective stress $\tilde{S}$.
Re-interpretation of strain equivalence for finite strains [present model]

The strain associated with a damaged state under the applied stress $\hat{S}$ is equivalent to the strain associated with the undamaged state under the effective stress $\tilde{S}$

\[
\hat{E} = D^d : \hat{S} \\
\tilde{E} = D^0 : \tilde{S}
\]
Re-interpretation of strain equivalence for finite strains [present model]

The strain associated with a damaged state under the applied stress $\hat{\mathbf{S}}$ is equivalent to the strain associated with the undamaged state under the effective stress $\tilde{\mathbf{S}}$

\[
\hat{\mathbf{E}} = \mathbb{D}^d : \hat{\mathbf{S}} \\
\tilde{\mathbf{E}} = \mathbb{D}^0 : \tilde{\mathbf{S}} \\
\hat{\mathbf{E}} = \tilde{\mathbf{E}}
\]
Re-interpretation of strain equivalence for finite strains [present model]

The strain associated with a damaged state under the applied stress $\hat{S}$ is equivalent to the strain associated with the undamaged state under the effective stress $\tilde{S}$

$$\hat{\mathbf{E}} = \mathbf{D}^d : \hat{\mathbf{S}}$$
$$\tilde{\mathbf{E}} = \mathbf{D}^0 : \tilde{\mathbf{S}}$$

$$\hat{\mathbf{E}} = \tilde{\mathbf{E}} \quad \Rightarrow \quad \tilde{\mathbf{S}} = \mathbf{C}^0 : \mathbf{D}^d : \hat{\mathbf{S}}$$
Re-interpretation of strain equivalence for finite strains [present model]

The strain associated with a damaged state under the applied stress $\hat{S}$ is equivalent to the strain associated with the undamaged state under the effective stress $\tilde{S}$

\[
\hat{E} = \mathbb{D}^d : \hat{S} \\
\tilde{E} = \mathbb{D}^0 : \tilde{S} \quad \Rightarrow \quad \tilde{S} = \mathbb{C}^0 : \mathbb{D}^d : \hat{S}
\]

We obtain:

\[
\tilde{S} = ((\lambda (6 \eta_2 + 2 \eta_4 + \eta_5) + \mu 4 \eta_2) \text{tr}((\hat{G} - \hat{D})^{-1}) \text{tr}(\hat{S}) \hat{G}^{-1} \\
+ (\lambda (3 \eta_5 + 2 \eta_6) + \mu 2 \eta_5) \text{tr}(\hat{S} (\hat{G} - \hat{D})^{-1}) \hat{G}^{-1} + \mu 2 \eta_5 \text{tr}(\hat{S}) (\hat{G} - \hat{D})^{-1} \\
+ \mu 4 \eta_4 \text{tr}((\hat{G} - \hat{D})^{-1}) \hat{S} + \mu 2 \eta_6 \left((\hat{G} - \hat{D})^{-1} \hat{G} \hat{S} + \hat{S} \hat{G} (\hat{G} - \hat{D})^{-1}\right)
\]
Re-interpretation of strain equivalence for finite strains [present model]

The strain associated with a damaged state under the applied stress $\hat{\mathbf{S}}$ is equivalent to the strain associated with the undamaged state under the effective stress $\tilde{\mathbf{S}}$

$$\hat{\mathbf{E}} = \mathbb{D}^d : \hat{\mathbf{S}}$$
$$\tilde{\mathbf{E}} = \mathbb{D}^0 : \tilde{\mathbf{S}}$$

We obtain:

$$\tilde{\mathbf{S}} = ((\lambda (6 \eta_2 + 2 \eta_4 + \eta_5) + \mu 4 \eta_2) \, \text{tr}((\hat{\mathbf{G}} - \hat{\mathbf{D}})^{-1}) \, \text{tr}(\hat{\mathbf{S}}) \, \hat{\mathbf{G}}^{-1}$$

$$+ (\lambda (3 \eta_5 + 2 \eta_6) + \mu 2 \eta_5) \, \text{tr}(\hat{\mathbf{S}}(\hat{\mathbf{G}} - \hat{\mathbf{D}})^{-1}) \, \hat{\mathbf{G}}^{-1} + \mu 2 \eta_5 \, \text{tr}(\hat{\mathbf{S}}) \, (\hat{\mathbf{G}} - \hat{\mathbf{D}})^{-1}$$

$$+ \mu 4 \eta_4 \, \text{tr}((\hat{\mathbf{G}} - \hat{\mathbf{D}})^{-1}) \, \hat{\mathbf{S}} + \mu 2 \eta_6 \left( (\hat{\mathbf{G}} - \hat{\mathbf{D}})^{-1} \hat{\mathbf{G}} \hat{\mathbf{S}} + \hat{\mathbf{S}} \hat{\mathbf{G}} (\hat{\mathbf{G}} - \hat{\mathbf{D}})^{-1} \right)$$

Finally, $\tilde{\mathbf{S}}$ is used as effective stress tensor in the inelastic potential:

$$F = \sqrt{\text{dev}(\tilde{\mathbf{C}} \tilde{\mathbf{S}} - \tilde{\mathbf{Y}}) \text{dev}(\tilde{\mathbf{C}} \tilde{\mathbf{S}} - \tilde{\mathbf{Y}}) : \mathbf{I} - \sqrt{\frac{2}{3}} \, q_1(\alpha)}$$

$$F = \sqrt{\text{dev}(\tilde{\mathbf{G}} (\tilde{\mathbf{S}} - \tilde{\kappa})) \text{dev}(\tilde{\mathbf{G}} (\tilde{\mathbf{S}} - \tilde{\kappa})) : \mathbf{I} - \sqrt{\frac{2}{3}} \, q_1(\alpha)} \quad \tilde{\mathbf{S}} = \hat{\mathbf{G}}^{-1} \hat{\mathbf{C}} \tilde{\mathbf{S}}$$

$$F = \sqrt{\text{dev}(\mathbf{g}(\tilde{\mathbf{S}} - \mathbf{\rho})) \text{dev}(\mathbf{g}(\tilde{\mathbf{S}} - \mathbf{\rho})) : \mathbf{i} - \sqrt{\frac{2}{3}} \, q_1(\alpha)} \quad \mathbf{\tau} = F^e \mathbf{\tau}(\tilde{\mathbf{S}})$$
Conclusion
Conclusion

- Equivalence principles are necessary for the small strain case. They lead, however, often to an anisotropic Cauchy stress tensor which requires a certain type of ad-hoc-symmetrization of this tensor. This seems to be arbitrary and not conclusive.
Conclusion

- Equivalence principles are necessary for the small strain case. They lead, however, often to an anisotropic Cauchy stress tensor which requires a certain type of ad-hoc-symmetrization of this tensor. This seems to be arbitrary and not conclusive.

- In the finite strain case the use of a fictive (undamaged) state can be introduced as a specific configuration which can be reached by proposing a certain multiplicative decomposition of the deformation gradient into an elasto-plastic and a (fictive) damage part.
Conclusion

- Equivalence principles are necessary for the small strain case. They lead, however, often to an anisotropic Cauchy stress tensor which requires a certain type of ad-hoc-symmetrization of this tensor. This seems to be arbitrary and not conclusive.
- In the finite strain case the use of a fictive (undamaged) state can be introduced as a specific configuration which can be reached by proposing a certain multiplicative decomposition of the deformation gradient into an elasto-plastic and a (fictive) damage part.
- The net stress concept, the energy equivalence principle and the strain equivalence principle can be simply extended to the finite strain case.
Conclusion

- Equivalence principles are necessary for the small strain case. They lead, however, often to an anisotropic Cauchy stress tensor which requires a certain type of ad-hoc-symmetrization of this tensor. This seems to be arbitrary and not conclusive.

- In the finite strain case the use of a fictive (undamaged) state can be introduced as a specific configuration which can be reached by proposing a certain multiplicative decomposition of the deformation gradient into an elasto-plastic and a (fictive) damage part.

- The net stress concept, the energy equivalence principle and the strain equivalence principle can be simply extended to the finite strain case.

- A rigorous treatment for finite strains leads automatically to a symmetrical effective stress tensor and, therefore, circumvents the problem of an ad-hoc-symmetrization like for small strains.
Conclusion

- Equivalence principles are necessary for the small strain case. They lead, however, often to an anisotropic Cauchy stress tensor which requires a certain type of ad-hoc-symmetrization of this tensor. This seems to be arbitrary and not conclusive.
- In the finite strain case the use of a fictive (undamaged) state can be introduced as a specific configuration which can be reached by proposing a certain multiplicative decomposition of the deformation gradient into an elasto-plastic and a (fictive) damage part.
- The net stress concept, the energy equivalence principle and the strain equivalence principle can be simply extended to the finite strain case.
- A rigorous treatment for finite strains leads automatically to a symmetrical effective stress tensor and, therefore, circumvents the problem of an ad-hoc-symmetrization like for small strains.
- The proposed model is a consistent extension of Lemaitre's model to the anisotropic finite strain case. The use of \((I - D)^{-1}\) as structural tensor is superior to those models where only \(D\) is used in the complementary stress energy function, since it is
  (1) highly nonlinear in \(D\) and
  (2) is tractable by means of ordinary differentiation rules in contrast to the use of a structural tensor \((I - D)^{-1/2}\) which requires a consideration in principal values.
Conclusion

- Equivalence principles are necessary for the small strain case. They lead, however, often to an anisotropic Cauchy stress tensor which requires a certain type of ad-hoc-symmetrization of this tensor. This seems to be arbitrary and not conclusive.

- In the finite strain case the use of a fictive (undamaged) state can be introduced as a specific configuration which can be reached by proposing a certain multiplicative decomposition of the deformation gradient into an elasto-plastic and a (fictive) damage part.

- The net stress concept, the energy equivalence principle and the strain equivalence principle can be simply extended to the finite strain case.

- A rigorous treatment for finite strains leads automatically to a symmetrical effective stress tensor and, therefore, circumvents the problem of an ad-hoc-symmetrization like for small strains.

- The proposed model is a consistent extension of Lemaitre's model to the anisotropic finite strain case. The use of $(\mathbf{I} - \mathbf{D})^{-1}$ as structural tensor is superior to those models where only $\mathbf{D}$ is used in the complementary stress energy function, since it is
  (1) highly nonlinear in $\mathbf{D}$ and
  (2) is tractable by means of ordinary differentiation rules in contrast to the use of a structural tensor $\left(\mathbf{I} - \mathbf{D}\right)^{-1/2}$ which requires a consideration in principal values.

- However, the model is only sketched so far, certain points should be worked out. E.g. the consequences of the convexity of the stress energy function should be analyzed and a consistent thermodynamic framework should be found out.