Foundations of tensor algebra and analysis

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1 Tensor algebra

Indices:

$$\alpha,\ \beta,\ \gamma,\ \delta,\ \ldots \in \{1,2\}$$
$$i,\ j,\ k,\ l,\ m,\ldots \in \{1,2,3\}$$

Kronecker delta:

$$\delta_{ij} = \delta^{ij} = \delta^i_j = \delta^j_i \begin{cases} 1 \text{ for } i = j \\ 0 \text{ for } i \neq j \end{cases} \quad .$$

Einstein summation convention:

If an index appears twice within a tensor component relation in co-variant and contra-variant position, we have to sum with respect to these indices. This index is called summation index (or dummy index) and is different from a free index which appears only once. Three-fold or four-fold appearing indices are not allowed.

Representation of tensors of first order (vectors):

$$\boldsymbol{A}^\flat = A_i \boldsymbol{G}^i \text{ (co-variant component)} \quad ,$$
$$\boldsymbol{A}^\sharp = A^i \boldsymbol{G}_i \text{ (contra-variant component)} \quad .$$

Remark: *In what follows, all tensors are represented with respect to the reference placement using the material basis vectors $\boldsymbol{G}_i$ and $\boldsymbol{G}^i$. A similar representation were imaginable with respect to the current placement ($\boldsymbol{g}_i$ and$\boldsymbol{g}^i$) or e.g. the intermediate placement($\hat{\boldsymbol{G}}_i$ and $\hat{\boldsymbol{G}}^i$).*

Addition of vectors:

$$\boldsymbol{A}^\flat \pm \boldsymbol{B}^\flat = A_i \boldsymbol{G}^i \pm B_j \boldsymbol{G}^j = (A_i \pm B_i)\boldsymbol{G}^i \quad .$$

Dual basis:

The basis vectors $\boldsymbol{G}_i$ and $\boldsymbol{G}^i$ are orthogonal to each other:

$$\boldsymbol{G}_i \cdot \boldsymbol{G}^j = \delta^j_i = \boldsymbol{G}^j \cdot \boldsymbol{G}_i \quad ,$$

whereby $(\cdot)$ is called scalar product.

Dyadic product of two vectors:

$$\boldsymbol{A}^\flat \otimes \boldsymbol{B}^\flat = (A_i \boldsymbol{G}^i) \otimes (B_j \boldsymbol{G}^j) = A_i B_j \boldsymbol{G}^i \otimes \boldsymbol{G}^j = A_{ij} \boldsymbol{G}^i \otimes \boldsymbol{G}^j$$

with $A_{ij} = A_i B_j$.

Metric tensor components:

$$G_{ij} = G_{ji} = \boldsymbol{G}_i \cdot \boldsymbol{G}_j = \boldsymbol{G}_j \cdot \boldsymbol{G}_i, \quad G^{ij} = G^{ji} = \boldsymbol{G}^i \cdot \boldsymbol{G}^j = \boldsymbol{G}^j \cdot \boldsymbol{G}^i,$$
$$G^{ij} G_{jk} = G_{kj} G^{ji} = \delta^i_k \quad .$$

Using the metric tensor components, the contra-variant (co-variant) basis vector can be transformed into a co-variant (contra-variant) basis vector :

$$\boldsymbol{G}_i = (\boldsymbol{G}_i \cdot \boldsymbol{G}_j) \boldsymbol{G}^j = G_{ij} \boldsymbol{G}^j, \quad \boldsymbol{G}^i = (\boldsymbol{G}^i \cdot \boldsymbol{G}^j) \boldsymbol{G}_j = G^{ij} \boldsymbol{G}_j \ .$$

Metric (Identity) tensors of the reference placement:

$$\begin{aligned} \mathbf{G} &= G_{ij} \boldsymbol{G}^i \otimes \boldsymbol{G}^j, \\ \mathbf{G}^{-1} &= G^{ij} \boldsymbol{G}_i \otimes \boldsymbol{G}_j, \\ \mathbf{I} &= \boldsymbol{G}_i \otimes \boldsymbol{G}^i, \\ \mathbf{I}^* &= \boldsymbol{G}^i \otimes \boldsymbol{G}_i. \end{aligned}$$

Raising and lowering of indices:

Using the metric tensor components, the contra-variant (co-variant) component can be transformed into a co-variant (contra-variant) component :

$$A_i = G_{ij} A^j, \qquad A^i = G^{ij} A_j.$$

In an absolute notation we can write these expressions as:

$$\begin{aligned} \boldsymbol{A}^\flat &= \mathbf{G} \boldsymbol{A}^\sharp && \text{(Lowering of index)}, \\ \boldsymbol{A}^\sharp &= \mathbf{G}^{-1} \boldsymbol{A}^\flat && \text{(Raising of index)}. \end{aligned}$$

Dot product of two vectors:

$$\boldsymbol{A}^\flat \cdot \boldsymbol{B}^\flat = A_i B_j G^{ij} = A_i B^i = A^j B_j = A^i B^j G_{ij}.$$

Symbolic distinction of the dot product into scalar and vector product:

$$\begin{aligned} &< \boldsymbol{A}^\flat, \boldsymbol{B}^\flat >_{X^*} = A_i B_j < \boldsymbol{G}^i, \boldsymbol{G}^j >_{X^*} = A_i B_j G^{ij}, && \text{(Vector product)} \\ &< \boldsymbol{A}^\sharp, \boldsymbol{B}^\sharp >_X = A^i B^j < \boldsymbol{G}_i, \boldsymbol{G}_j >_X = A^i B^j G_{ij}, && \text{(Vector product)} \\ &\boldsymbol{A}^\flat \cdot \boldsymbol{B}^\sharp = \boldsymbol{A}^\sharp \cdot \boldsymbol{B}^\flat. && \text{(Scalar product)} \end{aligned}$$

Transformation between scalar and vector product:

$$
\begin{aligned}
< \boldsymbol{A}^\flat, \boldsymbol{B}^\flat >_{X^*} &= \mathbf{G}^{-1}\boldsymbol{A}^\flat \cdot \boldsymbol{B}^\flat = \boldsymbol{A}^\flat \cdot \mathbf{G}^{-1}\boldsymbol{B}^\flat, \\
< \boldsymbol{A}^\sharp, \boldsymbol{B}^\sharp >_{X} &= \mathbf{G}\boldsymbol{A}^\sharp \cdot \boldsymbol{B}^\sharp \quad = \boldsymbol{A}^\sharp \cdot \mathbf{G}\boldsymbol{B}^\sharp.
\end{aligned}
$$

Representation of tensors of second order:

$$
\begin{aligned}
\mathbf{A}^\flat &= A_{ij}\boldsymbol{G}^i \otimes \boldsymbol{G}^j \text{ (co-co-variant)} \\
\mathbf{A}^\sharp &= A^{ij}\boldsymbol{G}_i \otimes \boldsymbol{G}_j \text{ (contra-contra-variant)} \\
\mathbf{A}^\backslash &= A^{i}_{.j}\boldsymbol{G}_i \otimes \boldsymbol{G}^j \text{ (contra-co-variant)} \\
\mathbf{A}^/ &= A_{i.}^{\ j}\boldsymbol{G}^i \otimes \boldsymbol{G}_j \text{ (co-contra-variant)}
\end{aligned}
$$

Remark: *A symbolic distinction of component variance (position of indices) is redundant if the component decomposition is already clear.*

The dual of a second-order tensor:

The dual is formed by exchanging the order of basis vectors within the dyadic product.

Example: $\mathbf{A}^* = (A^{i}_{.j}\boldsymbol{G}_i \otimes \boldsymbol{G}^j)^* = A^{i}_{.j}\boldsymbol{G}^j \otimes \boldsymbol{G}_i$

$$
\begin{aligned}
(\mathbf{A}^\flat)^* &= A_{ij}\boldsymbol{G}^j \otimes \boldsymbol{G}^i, \\
(\mathbf{A}^\sharp)^* &= A^{ij}\boldsymbol{G}_j \otimes \boldsymbol{G}_i, \\
(\mathbf{A}^\backslash)^* &= A^{i}_{.j}\boldsymbol{G}^j \otimes \boldsymbol{G}_i, \\
(\mathbf{A}^/)^* &= A_{i.}^{\ j}\boldsymbol{G}_j \otimes \boldsymbol{G}^i.
\end{aligned}
$$

The transpose of a second-order tensor:

$$
\begin{aligned}
(\mathbf{A}^\backslash)^T &= \mathbf{G}^{-1}(\mathbf{A}^\backslash)^*\mathbf{G}, \\
(\mathbf{A}^/)^T &= \mathbf{G}(\mathbf{A}^/)^*\mathbf{G}^{-1}.
\end{aligned}
$$

Remark: *The transpose is identical to the dual after raising and lowering of indices. Therefore, the component variance is the same as before. A transpose for co-co-variant or contra-contra-variant tensors has no use, but could be defined by:*

$$
\begin{aligned}
(\mathbf{A}^\flat)^T &= \mathbf{G}^{-1}(\mathbf{A}^\flat)^*\mathbf{G}^{-1}, \\
(\mathbf{A}^\sharp)^T &= \mathbf{G}(\mathbf{A}^\sharp)^*\mathbf{G}.
\end{aligned}
$$

Here, the component variance is different than before.

Inverse:

The inverse of a second-order tensor is defined by:

$$
\begin{aligned}
&\mathbf{A}^{\backslash}(\mathbf{A}^{-1})^{\backslash} = (\mathbf{A}^{-1})^{\backslash}\mathbf{A}^{\backslash} = \mathbf{I},\\
&\mathbf{A}^{/}(\mathbf{A}^{-1})^{/} = (\mathbf{A}^{-1})^{/}\mathbf{A}^{/} = \mathbf{I}^{*},\\
&\mathbf{A}^{\flat}(\mathbf{A}^{-1})^{\sharp} = \mathbf{I}^{*}, \qquad (\mathbf{A}^{-1})^{\sharp}\mathbf{A}^{\flat} = \mathbf{I},\\
&\mathbf{A}^{\sharp}(\mathbf{A}^{-1})^{\flat} = \mathbf{I}, \qquad (\mathbf{A}^{-1})^{\flat}\mathbf{A}^{\sharp} = \mathbf{I}^{*}.
\end{aligned}
$$

(Skew-)Symmetry of a second-order tensor:

For the definition of symmetry properties it is useful to consider tensors with equal component variance. Therefore, the definition of a (skew-)symmetric part of a tensor is given as follows:

$$
\begin{aligned}
(\mathbf{A}^{\flat})_{sym} &= \tfrac{1}{2}(\mathbf{A}^{\flat} + (\mathbf{A}^{\flat})^{*}), & (\mathbf{A}^{\flat})_{skw} &= \tfrac{1}{2}(\mathbf{A}^{\flat} - (\mathbf{A}^{\flat})^{*}),\\
(\mathbf{A}^{\sharp})_{sym} &= \tfrac{1}{2}(\mathbf{A}^{\sharp} + (\mathbf{A}^{\sharp})^{*}), & (\mathbf{A}^{\sharp})_{skw} &= \tfrac{1}{2}(\mathbf{A}^{\sharp} - (\mathbf{A}^{\sharp})^{*})\\
(\mathbf{A}^{\backslash})_{sym} &= \tfrac{1}{2}(\mathbf{A}^{\backslash} + (\mathbf{A}^{\backslash})^{T}), & (\mathbf{A}^{\backslash})_{skw} &= \tfrac{1}{2}(\mathbf{A}^{\backslash} - (\mathbf{A}^{\backslash})^{T})\\
(\mathbf{A}^{/})_{sym} &= \tfrac{1}{2}(\mathbf{A}^{/} + (\mathbf{A}^{/})^{T}), & (\mathbf{A}^{/})_{skw} &= \tfrac{1}{2}(\mathbf{A}^{/} - (\mathbf{A}^{/})^{T}).
\end{aligned}
$$

Trace of a second-order tensor of order n:

$$
\begin{aligned}
&\mathrm{tr}(\mathbf{A}^{\flat})^{n} = (\mathbf{A}^{\flat}\mathbf{G}^{-1})^{n} : \mathbf{I},\\
&\mathrm{tr}(\mathbf{A}^{\sharp})^{n} = (\mathbf{A}^{\sharp}\mathbf{G})^{n} : \mathbf{I}^{*},\\
&\mathrm{tr}(\mathbf{A}^{\backslash})^{n} = (\mathbf{A}^{\backslash})^{n} : \mathbf{I}^{*},\\
&\mathrm{tr}(\mathbf{A}^{/})^{n} = (\mathbf{A}^{/})^{n} : \mathbf{I}.
\end{aligned}
$$

Deviatoric and spherical part of a second-order tensor:

$$
\begin{aligned}
\mathbf{A}^{\flat}_{dev} &= \mathbf{A}^{\flat} - \tfrac{1}{3}\,\mathrm{tr}(\mathbf{A}^{\flat})\,\mathbf{G}, & \mathbf{A}^{\flat}_{sph} &= \tfrac{1}{3}\,\mathrm{tr}(\mathbf{A}^{\flat})\,\mathbf{G},\\
\mathbf{A}^{\sharp}_{dev} &= \mathbf{A}^{\sharp} - \tfrac{1}{3}\,\mathrm{tr}(\mathbf{A}^{\sharp})\,\mathbf{G}^{-1}, & \mathbf{A}^{\sharp}_{sph} &= \tfrac{1}{3}\,\mathrm{tr}(\mathbf{A}^{\sharp})\,\mathbf{G}^{-1},\\
\mathbf{A}^{\backslash}_{dev} &= \mathbf{A}^{\backslash} - \tfrac{1}{3}\,\mathrm{tr}(\mathbf{A}^{\backslash})\,\mathbf{I}, & \mathbf{A}^{\backslash}_{sph} &= \tfrac{1}{3}\,\mathrm{tr}(\mathbf{A}^{\backslash})\,\mathbf{I},\\
\mathbf{A}^{/}_{dev} &= \mathbf{A}^{/} - \tfrac{1}{3}\,\mathrm{tr}(\mathbf{A}^{/})\,\mathbf{I}^{*}, & \mathbf{A}^{/}_{sph} &= \tfrac{1}{3}\,\mathrm{tr}(\mathbf{A}^{/})\,\mathbf{I}^{*}.
\end{aligned}
$$

Addition of tensors of second order:

$$
\begin{aligned}
&\mathbf{A}^{\flat} \pm \mathbf{B}^{\flat} = (A_{ij} \pm B_{ij})\; \boldsymbol{G}^{i} \otimes \boldsymbol{G}^{j},\\
&\mathbf{A}^{\sharp} \pm \mathbf{B}^{\sharp} = (A^{ij} \pm B^{ij})\; \boldsymbol{G}_{i} \otimes \boldsymbol{G}_{j},\\
&\mathbf{A}^{\backslash} \pm \mathbf{B}^{\backslash} = (A^{i}_{\cdot j} \pm B^{i}_{\cdot j})\, \boldsymbol{G}_{i} \otimes \boldsymbol{G}^{j},\\
&\mathbf{A}^{/} \pm \mathbf{B}^{/} = (A_{i\cdot}^{\;j} \pm B_{i\cdot}^{\;j})\, \boldsymbol{G}^{i} \otimes \boldsymbol{G}_{j}.
\end{aligned}
$$

Simple contraction of tensors of first and second order:

E.g.:

$$\mathbf{A}^\flat \boldsymbol{A}^\sharp = (A_{ij}\boldsymbol{G}^i \otimes \boldsymbol{G}^j)\cdot(A^k\boldsymbol{G}_k) = A_{ij}A^j\boldsymbol{G}^i,$$
$$\boldsymbol{A}^\flat \mathbf{A}^\sharp = (A_k\boldsymbol{G}^k)\cdot(A^{ij}\boldsymbol{G}_i \otimes \boldsymbol{G}_j) = A^{kj}A_k\boldsymbol{G}_j,$$
$$\mathbf{A}^\flat \mathbf{B}^\sharp = (A_{ij}\boldsymbol{G}^i \otimes \boldsymbol{G}^j)\cdot(B^{kl}\boldsymbol{G}_k \otimes \boldsymbol{G}_l) = A_{ik}B^{kl}\boldsymbol{G}^i \otimes \boldsymbol{G}_l.$$

Double contraction of tensors of second-order:

$$\mathbf{A}^\flat : \mathbf{B}^\sharp = A_{ij}B^{ij},$$
$$\mathbf{A}^\sharp : \mathbf{B}^\flat = A^{ij}B_{ij},$$
$$\mathbf{A}^\backslash : \mathbf{B}^/ = A^i_{.j}B_i{}^j_{.},$$
$$\mathbf{A}^/ : \mathbf{B}^\backslash = A_i{}^j_{.}B^i_{.j}.$$

Representation of tensors of fourth order:

$$\mathbb{E}^{\sharp\sharp} = E^{ijkl}\boldsymbol{G}_i \otimes \boldsymbol{G}_j \otimes \boldsymbol{G}_k \otimes \boldsymbol{G}_l,$$
$$\mathbb{E}^{\flat\flat} = E_{ijkl}\boldsymbol{G}^i \otimes \boldsymbol{G}^j \otimes \boldsymbol{G}^k \otimes \boldsymbol{G}^l.$$

Remark: *Depending on the position of indices there exist 14 additional component decompositions for a fourth-order tensor.*

Tensor products for a tensor of fourth order:

$$\mathbb{E}^{\flat\flat}_{ijkl} = (\mathbf{A}^\flat \otimes \mathbf{B}^\flat)_{ijkl} = A_{ij}B_{kl},$$
$$\mathbb{E}^{\flat\flat}_{ijkl} = (\mathbf{A}^\flat \times \mathbf{B}^\flat)_{ijkl} = A_{il}B_{jk},$$
$$\mathbb{E}^{\flat\flat}_{ijkl} = (\mathbf{A}^\flat \boxtimes \mathbf{B}^\flat)_{ijkl} = A_{ik}B_{jl}.$$

Simple contractions of tensors of second and fourth order:

$$\mathbb{E}^{\flat\backslash}\mathbf{A}^\sharp = E_{ij}{}^k_{.\,m}A^{ml}\boldsymbol{G}^i \otimes \boldsymbol{G}^j \otimes \boldsymbol{G}_k \otimes \boldsymbol{G}_l,$$
$$\mathbf{A}^\backslash\mathbb{E}^{\sharp\sharp} = A^i_{.m}E^{mjkl}\boldsymbol{G}_i \otimes \boldsymbol{G}_j \otimes \boldsymbol{G}_k \otimes \boldsymbol{G}_l.$$

Double contractions of tensors of fourth order:

$$\mathbb{E}^{\flat\flat} : \mathbb{D}^{\sharp\flat} = E_{ijmn}D^{mn}_{..\,kl}\boldsymbol{G}^i \otimes \boldsymbol{G}^j \otimes \boldsymbol{G}^k \otimes \boldsymbol{G}^l,$$
$$\mathbb{E}^{\flat\flat}\bullet\circ\mathbb{D}^{\sharp\sharp} = E_{imnj}D^{mkln}\,\boldsymbol{G}^i \otimes \boldsymbol{G}_k \otimes \boldsymbol{G}_l \otimes \boldsymbol{G}^j,$$
$$\mathbb{E}^{\flat\flat}\circ\bullet\mathbb{D}^{\sharp\sharp} = E_{mjkn}D^{imnl}\,\boldsymbol{G}_i \otimes \boldsymbol{G}^j \otimes \boldsymbol{G}^k \otimes \boldsymbol{G}_l.$$

Double contractions of tensors of second and fourth order:

$$\mathbf{A}^\sharp : \mathbb{E}^{\flat\flat} = A^{mn} E_{mnij} \boldsymbol{G}^i \otimes \boldsymbol{G}^j, \quad \mathbb{E}^{\flat\flat} : \mathbf{A}^\sharp = E_{ijmn} A^{mn} \boldsymbol{G}^i \otimes \boldsymbol{G}^j,$$

$$\mathbf{A}^\sharp \bullet\!\circ \mathbb{E}^{\flat\flat} = A^{mn} E_{mijn} \boldsymbol{G}^i \otimes \boldsymbol{G}^j, \quad \mathbb{E}^{\flat\flat} \bullet\!\circ \mathbf{A}^\sharp = E_{imnj} A^{mn} \boldsymbol{G}^i \otimes \boldsymbol{G}^j,$$

$$\mathbf{A}^\sharp \circ\!\bullet \mathbb{E}^{\flat\flat} = A^{mn} E_{imnj} \boldsymbol{G}^i \otimes \boldsymbol{G}^j, \quad \mathbb{E}^{\flat\flat} \circ\!\bullet \mathbf{A}^\sharp = E_{mijn} A^{mn} \boldsymbol{G}^i \otimes \boldsymbol{G}^j.$$

For second-order tensors the distinction of inner and outer bases is meaningless:

$$\mathbf{A}^\flat : \mathbf{B}^\sharp = \mathbf{A}^\flat \bullet\!\circ \mathbf{B}^\sharp = \mathbf{A}^\flat \circ\!\bullet \mathbf{B}^\sharp.$$

Transposition operations for fourth-order tensors:

$$E_{ijkl} \to E_{jikl} \Longrightarrow \mathbb{E}^{\flat\flat} \to (\mathbb{E}^{\flat\flat})^{dl},$$

$$E_{ijkl} \to E_{ijlk} \Longrightarrow \mathbb{E}^{\flat\flat} \to (\mathbb{E}^{\flat\flat})^{dr},$$

$$E_{ijkl} \to E_{jilk} \Longrightarrow \mathbb{E}^{\flat\flat} \to (\mathbb{E}^{\flat\flat})^{d},$$

$$E_{ijkl} \to E_{klij} \Longrightarrow \mathbb{E}^{\flat\flat} \to (\mathbb{E}^{\flat\flat})^{D}.$$

$$E_{ijkl} \to E_{ikjl} \Longrightarrow \mathbb{E}^{\flat\flat} \to (\mathbb{E}^{\flat\flat})^{ti},$$

$$E_{ijkl} \to E_{ljki} \Longrightarrow \mathbb{E}^{\flat\flat} \to (\mathbb{E}^{\flat\flat})^{to},$$

$$E_{ijkl} \to E_{lkji} \Longrightarrow \mathbb{E}^{\flat\flat} \to (\mathbb{E}^{\flat\flat})^{t},$$

$$E_{ijkl} \to E_{jilk} \Longrightarrow \mathbb{E}^{\flat\flat} \to (\mathbb{E}^{\flat\flat})^{T}.$$

Symmetry-properties of a fourth-order tensor:

A tensor $\mathbb{E}$ fulfills minor symmetry if:
$$\mathbb{E} = \mathbb{E}^{ti} = \mathbb{E}^{to} \text{ or } \mathbb{E} = \mathbb{E}^{dl} = \mathbb{E}^{dr}.$$

A tensor $\mathbb{E}$ fulfills major symmetry if:
$$\mathbb{E} = \mathbb{E}^{T} \text{ or } \mathbb{E} = \mathbb{E}^{D}.$$

A tensor $\mathbb{E}$ is supersymmetric if:
$$\mathbb{E} = \mathbb{E}^{ti} = \mathbb{E}^{to} = \mathbb{E}^{T} \text{ or } \mathbb{E} = \mathbb{E}^{dl} = \mathbb{E}^{dr} = \mathbb{E}^{D}.$$

In particular, considering the tensors $\mathbb{E} = \mathbb{G}{\bullet}{\circ}\mathbb{C}{\bullet}{\circ}\mathbb{H}$ and $\mathbb{F} = \mathbb{K} : \mathbb{D} : \mathbb{L}$, the following applies:

$$
\begin{aligned}
\mathbb{E}^T &= \mathbb{H}^T{\bullet}{\circ}\mathbb{C}^T{\bullet}{\circ}\mathbb{G}^T \ , & \mathbb{F}^D &= \mathbb{L}^D : \mathbb{D}^D : \mathbb{K}^D \ , \\
\mathbb{E}^{ti} &= \mathbb{G}{\bullet}{\circ}\mathbb{C}{\bullet}{\circ}\mathbb{H}^{ti} \ , & \mathbb{F}^{dr} &= \mathbb{K} : \mathbb{D} : \mathbb{L}^{dr} \ , \\
\mathbb{E}^{to} &= \mathbb{G}^{to}{\bullet}{\circ}\mathbb{C}{\bullet}{\circ}\mathbb{H} \ , & \mathbb{F}^{dl} &= \mathbb{K}^{dl} : \mathbb{D} : \mathbb{L} \ .
\end{aligned}
$$

2 Tensor analysis

Tensor differentiation in absolute notation:

$$
\begin{aligned}
\frac{\partial f}{\partial \mathbf{X}^\flat} &= \frac{\partial f}{\partial \mathbf{X}_{ij}} \boldsymbol{G}_i \otimes \boldsymbol{G}_j, \\
\frac{\partial f}{\partial \mathbf{X}^\sharp} &= \frac{\partial f}{\partial \mathbf{X}^{ij}} \boldsymbol{G}^i \otimes \boldsymbol{G}^j, \\
\frac{\partial f}{\partial \mathbf{X}^\backslash} &= \frac{\partial f}{\partial \mathbf{X}^i_{\cdot j}} \boldsymbol{G}^i \otimes \boldsymbol{G}_j, \\
\frac{\partial f}{\partial \mathbf{X}^/} &= \frac{\partial f}{\partial \mathbf{X}_i{}^j_{\cdot}} \boldsymbol{G}_i \otimes \boldsymbol{G}^j, \\
\frac{\partial \mathbf{F}^\flat}{\partial \mathbf{X}^\flat} &= \frac{\partial F_{ij}}{\partial X_{kl}} \boldsymbol{G}^i \otimes \boldsymbol{G}_k \otimes \boldsymbol{G}_l \otimes \boldsymbol{G}^j, \\
\frac{\partial \mathbf{F}^\flat}{\partial \mathbf{X}^\sharp} &= \frac{\partial F_{ij}}{\partial X^{kl}} \boldsymbol{G}^i \otimes \boldsymbol{G}^k \otimes \boldsymbol{G}^l \otimes \boldsymbol{G}^j, \\
\frac{\partial \mathbf{F}^\flat}{\partial \mathbf{X}^\backslash} &= \frac{\partial F_{ij}}{\partial X^k_{\cdot l}} \boldsymbol{G}^i \otimes \boldsymbol{G}^k \otimes \boldsymbol{G}_l \otimes \boldsymbol{G}^j, \\
\frac{\partial \mathbf{F}^\flat}{\partial \mathbf{X}^/} &= \frac{\partial F_{ij}}{\partial X_k{}^l_{\cdot}} \boldsymbol{G}^i \otimes \boldsymbol{G}_k \otimes \boldsymbol{G}^l \otimes \boldsymbol{G}^j, \\
\frac{\partial f\, \mathbf{F}^\flat}{\partial \mathbf{X}^\sharp} &= f\, \frac{\partial \mathbf{F}^\flat}{\partial \mathbf{X}^\sharp} + \mathbf{F}^\flat \times \frac{\partial f}{\partial \mathbf{X}^\sharp}, \\
\frac{\partial \mathbf{A}^\sharp \mathbf{B}^\flat}{\partial \mathbf{X}^\flat} &= \frac{\partial \mathbf{A}^\sharp}{\partial \mathbf{X}^\flat} \mathbf{B}^\flat + \mathbf{A}^\sharp \frac{\partial \mathbf{B}^\flat}{\partial \mathbf{X}^\flat}, \\
\frac{\partial \mathbf{A}^\flat : \mathbf{B}^\sharp}{\partial \mathbf{X}^\sharp} &= \frac{\partial \mathbf{A}^\flat}{\partial \mathbf{X}^\sharp} {\circ}{\bullet} \mathbf{B}^\sharp + \mathbf{A}^\flat {\bullet}{\circ} \frac{\partial \mathbf{B}^\sharp}{\partial \mathbf{X}^\sharp}.
\end{aligned}
$$

Special rules:

$$E_{i\cdot\cdot\,j}^{\;\,kl} = \frac{\partial A_{ij}}{\partial A_{kl}} = \delta_i^k \delta_j^l \Rightarrow \mathbb{E}^{/\backslash} = \frac{\partial \mathbf{A}^\flat}{\partial \mathbf{A}^\flat} = \mathbf{I}^* \otimes \mathbf{I},$$

$$E_{i\cdot\cdot\,j}^{\;\,kl} = \frac{\partial A_{ij}}{\partial A_{lk}} = \delta_i^l \delta_j^k \Rightarrow \mathbb{E}^{/\backslash} = \frac{\partial \mathbf{A}^\flat}{\partial (\mathbf{A}^\flat)^*} = \mathbf{I}^* \boxtimes \mathbf{I} = (\mathbf{I}^* \otimes \mathbf{I})^{ti},$$

$$E_{i\cdot\cdot\,j}^{\;\,kl} = \frac{\partial A_{ji}}{\partial A_{kl}} = \delta_j^k \delta_i^l \Rightarrow \mathbb{E}^{/\backslash} = \frac{\partial (\mathbf{A}^\flat)^*}{\partial \mathbf{A}^\flat} = \mathbf{I}^* \boxtimes \mathbf{I} = (\mathbf{I}^* \otimes \mathbf{I})^{to},$$

$$E_{i\cdot\cdot\,j}^{\;\,kl} = \frac{\partial A_{ji}}{\partial A_{lk}} = \delta_j^l \delta_i^k \Rightarrow \mathbb{E}^{/\backslash} = \frac{\partial (\mathbf{A}^\flat)^*}{\partial (\mathbf{A}^\flat)^*} = \mathbf{I}^* \otimes \mathbf{I} = (\mathbf{I}^* \otimes \mathbf{I})^{t}.$$

$$E_{\cdot\,kl}^{i\;\;\;\,j} = \frac{\partial A^{ij}}{\partial A^{kl}} = \delta_k^i \delta_l^j \Rightarrow \mathbb{E}^{\backslash/} = \frac{\partial \mathbf{A}^\sharp}{\partial \mathbf{A}^\sharp} = \mathbf{I} \otimes \mathbf{I}^*,$$

$$E_{\cdot\,kl}^{i\;\;\;\,j} = \frac{\partial A^{ij}}{\partial A^{lk}} = \delta_l^i \delta_k^j \Rightarrow \mathbb{E}^{\backslash/} = \frac{\partial \mathbf{A}^\sharp}{\partial (\mathbf{A}^\sharp)^*} = \mathbf{I} \boxtimes \mathbf{I}^* = (\mathbf{I} \otimes \mathbf{I}^*)^{ti},$$

$$E_{\cdot\,kl}^{i\;\;\;\,j} = \frac{\partial A^{ji}}{\partial A^{kl}} = \delta_k^j \delta_l^i \Rightarrow \mathbb{E}^{\backslash/} = \frac{\partial (\mathbf{A}^\sharp)^*}{\partial \mathbf{A}^\sharp} = \mathbf{I} \boxtimes \mathbf{I}^* = (\mathbf{I} \otimes \mathbf{I}^*)^{to},$$

$$E_{\cdot\,kl}^{i\;\;\;\,j} = \frac{\partial A^{ji}}{\partial A^{lk}} = \delta_l^j \delta_k^i \Rightarrow \mathbb{E}^{\backslash/} = \frac{\partial (\mathbf{A}^\sharp)^*}{\partial (\mathbf{A}^\sharp)^*} = \mathbf{I} \otimes \mathbf{I}^* = (\mathbf{I} \otimes \mathbf{I}^*)^{t}.$$

$$E_{\cdot\,k\,\cdot\,j}^{i\;\;\,l} = \frac{\partial A^i_{\cdot j}}{\partial A^k_{\cdot l}} = \delta_k^i \delta_j^l \Rightarrow \mathbb{E}^{\backslash\backslash} = \frac{\partial \mathbf{A}^\backslash}{\partial \mathbf{A}^\backslash} = \mathbf{I} \otimes \mathbf{I},$$

$$E_{\cdot\,\cdot\,kj}^{il} = \frac{\partial A^i_{\cdot j}}{\partial A^k_{\cdot l}} = \delta_k^i \delta_j^l \Rightarrow \mathbb{E}^{\sharp\flat} = \frac{\partial \mathbf{A}^\backslash}{\partial (\mathbf{A}^\backslash)^*} = \mathbf{I} \boxtimes \mathbf{I} = (\mathbf{I} \otimes \mathbf{I})^{ti},$$

$$E_{jk\,\cdot\,\cdot}^{\;\;\;\;li} = \frac{\partial A^i_{\cdot j}}{\partial A^k_{\cdot l}} = \delta_k^i \delta_j^l \Rightarrow \mathbb{E}^{\flat\sharp} = \frac{\partial (\mathbf{A}^\backslash)^*}{\partial \mathbf{A}^\backslash} = \mathbf{I}^* \boxtimes \mathbf{I}^* = (\mathbf{I} \otimes \mathbf{I})^{to},$$

$$E_{j\,\cdot\,k\,\cdot}^{\;\;l\;\;\;i} = \frac{\partial A^i_{\cdot j}}{\partial A^k_{\cdot l}} = \delta_k^i \delta_j^l \Rightarrow \mathbb{E}^{//} = \frac{\partial (\mathbf{A}^\backslash)^*}{\partial (\mathbf{A}^\backslash)^*} = \mathbf{I}^* \otimes \mathbf{I}^* = (\mathbf{I} \otimes \mathbf{I})^{t}.$$

$$E_{i\,\cdot\,l\,\cdot}^{\;\;k\;\;\;j} = \frac{\partial A_i{}^j_{\cdot}}{\partial A_k{}^l_{\cdot}} = \delta_i^k \delta_l^j \Rightarrow \mathbb{E}^{//} = \frac{\partial \mathbf{A}^/}{\partial \mathbf{A}^/} = \mathbf{I}^* \otimes \mathbf{I}^*,$$

$$E_{il\,\cdot\,\cdot}^{\;\;\;\;kj} = \frac{\partial A_i{}^j_{\cdot}}{\partial A_k{}^l_{\cdot}} = \delta_i^k \delta_l^j \Rightarrow \mathbb{E}^{\flat\sharp} = \frac{\partial \mathbf{A}^/}{\partial (\mathbf{A}^/)^*} = \mathbf{I}^* \boxtimes \mathbf{I}^* = (\mathbf{I}^* \otimes \mathbf{I}^*)^{ti},$$

$$E_{\cdot\,\cdot\,li}^{jk} = \frac{\partial A_i{}^j_{\cdot}}{\partial A_k{}^l_{\cdot}} = \delta_i^k \delta_l^j \Rightarrow \mathbb{E}^{\sharp\flat} = \frac{\partial (\mathbf{A}^/)^*}{\partial \mathbf{A}^/} = \mathbf{I} \boxtimes \mathbf{I} = (\mathbf{I}^* \otimes \mathbf{I}^*)^{to},$$

$$E_{\cdot\,l\,\cdot\,i}^{j\;\;k} = \frac{\partial A_i{}^j_{\cdot}}{\partial A_k{}^l_{\cdot}} = \delta_i^k \delta_l^j \Rightarrow \mathbb{E}^{\backslash\backslash} = \frac{\partial (\mathbf{A}^/)^*}{\partial (\mathbf{A}^/)^*} = \mathbf{I} \otimes \mathbf{I} = (\mathbf{I}^* \otimes \mathbf{I}^*)^{t}.$$

Differentiation with respect to a (skew-)symmetrical tensor:

$$\begin{aligned}
\frac{\partial \mathbf{F}^\flat}{\partial \mathbf{X}^\flat}\Big|_{\mathbf{X}^\flat=(\mathbf{X}^\flat)^*} &= \frac{\partial \mathbf{F}^\flat}{\partial \mathbf{X}^\flat}\bullet\!\circ \frac{1}{2}\,(\frac{\partial \mathbf{X}^\flat}{\partial \mathbf{X}^\flat}+\frac{\partial \mathbf{X}^\flat}{\partial (\mathbf{X}^\flat)^*})\\
&= \frac{\partial \mathbf{F}^\flat}{\partial \mathbf{X}^\flat}\bullet\!\circ \frac{1}{2}\,(\mathbf{I}^*\otimes \mathbf{I}+\mathbf{I}^*\boxtimes \mathbf{I}) = \frac{\partial \mathbf{F}^\flat}{\partial \mathbf{X}^\flat}\bullet\!\circ \mathbb{S}^{/\backslash},\\
\frac{\partial \mathbf{F}^\flat}{\partial \mathbf{X}^\sharp}\Big|_{\mathbf{X}^\sharp=(\mathbf{X}^\sharp)^*} &= \frac{\partial \mathbf{F}^\flat}{\partial \mathbf{X}^\sharp}\bullet\!\circ \frac{1}{2}\,(\frac{\partial \mathbf{X}^\sharp}{\partial \mathbf{X}^\sharp}+\frac{\partial \mathbf{X}^\sharp}{\partial (\mathbf{X}^\sharp)^*})\\
&= \frac{\partial \mathbf{F}^\flat}{\partial \mathbf{X}^\sharp}\bullet\!\circ \frac{1}{2}\,(\mathbf{I}\otimes \mathbf{I}^*+\mathbf{I}\boxtimes \mathbf{I}^*) = \frac{\partial \mathbf{F}^\flat}{\partial \mathbf{X}^\sharp}\bullet\!\circ \mathbb{S}^{\backslash /},\\
\frac{\partial \mathbf{F}^\flat}{\partial \mathbf{X}^\backslash}\Big|_{\mathbf{X}^\backslash=(\mathbf{X}^\backslash)^T} &= \frac{\partial \mathbf{F}^\flat}{\partial \mathbf{X}^\backslash}\bullet\!\circ \frac{1}{2}\,(\frac{\partial \mathbf{X}^\backslash}{\partial \mathbf{X}^\backslash}+\frac{\partial \mathbf{X}^\backslash}{\partial (\mathbf{X}^\backslash)^T})\\
&= \frac{\partial \mathbf{F}^\flat}{\partial \mathbf{X}^\backslash}\bullet\!\circ \frac{1}{2}\,(\mathbf{I}\otimes \mathbf{I}+\mathbf{G}^{-1}\boxtimes \mathbf{G}) = \frac{\partial \mathbf{F}^\flat}{\partial \mathbf{X}^\backslash}\bullet\!\circ \mathbb{S}^{\backslash\backslash},\\
\frac{\partial \mathbf{F}^\flat}{\partial \mathbf{X}^/}\Big|_{\mathbf{X}^/=(\mathbf{X}^/)^T} &= \frac{\partial \mathbf{F}^\flat}{\partial \mathbf{X}^/}\bullet\!\circ \frac{1}{2}\,(\frac{\partial \mathbf{X}^/}{\partial \mathbf{X}^/}+\frac{\partial \mathbf{X}^/}{\partial (\mathbf{X}^/)^T})\\
&= \frac{\partial \mathbf{F}^\flat}{\partial \mathbf{X}^/}\bullet\!\circ \frac{1}{2}\,(\mathbf{I}^*\otimes \mathbf{I}^*+\mathbf{G}\boxtimes \mathbf{G}^{-1}) = \frac{\partial \mathbf{F}^\flat}{\partial \mathbf{X}^/}\bullet\!\circ \mathbb{S}^{//}.
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \mathbf{F}^\flat}{\partial \mathbf{X}^\flat}\Big|_{\mathbf{X}^\flat=-(\mathbf{X}^\flat)^*} &= \frac{\partial \mathbf{F}^\flat}{\partial \mathbf{X}^\flat}\bullet\!\circ \frac{1}{2}\,(\frac{\partial \mathbf{X}^\flat}{\partial \mathbf{X}^\flat}-\frac{\partial \mathbf{X}^\flat}{\partial (\mathbf{X}^\flat)^*})\\
&= \frac{\partial \mathbf{F}^\flat}{\partial \mathbf{X}^\flat}\bullet\!\circ \frac{1}{2}\,(\mathbf{I}^*\otimes \mathbf{I}-\mathbf{I}^*\boxtimes \mathbf{I}) = \frac{\partial \mathbf{F}^\flat}{\partial \mathbf{X}^\flat}\bullet\!\circ \mathbb{A}^{/\backslash},\\
\frac{\partial \mathbf{F}^\flat}{\partial \mathbf{X}^\sharp}\Big|_{\mathbf{X}^\sharp=-(\mathbf{X}^\sharp)^*} &= \frac{\partial \mathbf{F}^\flat}{\partial \mathbf{X}^\sharp}\bullet\!\circ \frac{1}{2}\,(\frac{\partial \mathbf{X}^\sharp}{\partial \mathbf{X}^\sharp}-\frac{\partial \mathbf{X}^\sharp}{\partial (\mathbf{X}^\sharp)^*})\\
&= \frac{\partial \mathbf{F}^\flat}{\partial \mathbf{X}^\sharp}\bullet\!\circ \frac{1}{2}\,(\mathbf{I}\otimes \mathbf{I}^*-\mathbf{I}\boxtimes \mathbf{I}^*) = \frac{\partial \mathbf{F}^\flat}{\partial \mathbf{X}^\sharp}\bullet\!\circ \mathbb{A}^{\backslash /},\\
\frac{\partial \mathbf{F}^\flat}{\partial \mathbf{X}^\backslash}\Big|_{\mathbf{X}^\backslash=-(\mathbf{X}^\backslash)^T} &= \frac{\partial \mathbf{F}^\flat}{\partial \mathbf{X}^\backslash}\bullet\!\circ \frac{1}{2}\,(\frac{\partial \mathbf{X}^\backslash}{\partial \mathbf{X}^\backslash}-\frac{\partial \mathbf{X}^\backslash}{\partial (\mathbf{X}^\backslash)^T})\\
&= \frac{\partial \mathbf{F}^\flat}{\partial \mathbf{X}^\backslash}\bullet\!\circ \frac{1}{2}\,(\mathbf{I}\otimes \mathbf{I}-\mathbf{G}^{-1}\boxtimes \mathbf{G}) = \frac{\partial \mathbf{F}^\flat}{\partial \mathbf{X}^\backslash}\bullet\!\circ \mathbb{A}^{\backslash\backslash},\\
\frac{\partial \mathbf{F}^\flat}{\partial \mathbf{X}^/}\Big|_{\mathbf{X}^/=-(\mathbf{X}^/)^T} &= \frac{\partial \mathbf{F}^\flat}{\partial \mathbf{X}^/}\bullet\!\circ \frac{1}{2}\,(\frac{\partial \mathbf{X}^/}{\partial \mathbf{X}^/}-\frac{\partial \mathbf{X}^/}{\partial (\mathbf{X}^/)^T})\\
&= \frac{\partial \mathbf{F}^\flat}{\partial \mathbf{X}^/}\bullet\!\circ \frac{1}{2}\,(\mathbf{I}^*\otimes \mathbf{I}^*-\mathbf{G}\boxtimes \mathbf{G}^{-1}) = \frac{\partial \mathbf{F}^\flat}{\partial \mathbf{X}^/}\bullet\!\circ \mathbb{A}^{//}.
\end{aligned}$$

Differentiation of the inverse:

$$\frac{\partial(\mathbf{X}^{-1})^\sharp}{\partial\mathbf{X}^\flat} = -(\mathbf{X}^{-1})^\sharp \otimes (\mathbf{X}^{-1})^\sharp,$$

$$\frac{\partial(\mathbf{X}^{-1})^\flat}{\partial\mathbf{X}^\sharp} = -(\mathbf{X}^{-1})^\flat \otimes (\mathbf{X}^{-1})^\flat,$$

$$\frac{\partial(\mathbf{X}^{-1})^\backslash}{\partial\mathbf{X}^\backslash} = -(\mathbf{X}^{-1})^\backslash \otimes (\mathbf{X}^{-1})^\backslash,$$

$$\frac{\partial(\mathbf{X}^{-1})^/}{\partial\mathbf{X}^/} = -(\mathbf{X}^{-1})^/ \otimes (\mathbf{X}^{-1})^/.$$

Differentiation with respect to the inverse:

$$\frac{\partial\mathbf{X}^\flat}{\partial(\mathbf{X}^{-1})^\sharp} = -\mathbf{X}^\flat \otimes \mathbf{X}^\flat,$$

$$\frac{\partial\mathbf{X}^\sharp}{\partial(\mathbf{X}^{-1})^\flat} = -\mathbf{X}^\sharp \otimes \mathbf{X}^\sharp,$$

$$\frac{\partial\mathbf{X}^\backslash}{\partial(\mathbf{X}^{-1})^\backslash} = -\mathbf{X}^\backslash \otimes \mathbf{X}^\backslash,$$

$$\frac{\partial\mathbf{X}^/}{\partial(\mathbf{X}^{-1})^/} = -\mathbf{X}^/ \otimes \mathbf{X}^/.$$

Using the chain rule and product rule of differential calculus:

Using the above rules, we may state :

$$\begin{aligned}\Delta(\mathbf{A}^\flat\mathbf{B}^\sharp(\mathbf{X}^\flat)) = \frac{\partial\mathbf{A}^\flat\mathbf{B}^\sharp}{\partial\mathbf{X}^\flat}\bullet\circ\Delta\mathbf{X}^\flat &= (\mathbf{A}^\flat\mathbf{B}^\sharp)_{,\mathbf{X}^\flat}\bullet\circ\Delta\mathbf{X}^\flat \\ &= (\mathbf{A}^\flat{}_{,\mathbf{X}^\flat}\mathbf{B}^\sharp + \mathbf{A}^\flat\mathbf{B}^\sharp{}_{,\mathbf{X}^\flat})\bullet\circ\Delta\mathbf{X}^\flat.\end{aligned}$$

Using the contraction rule ($\bullet\circ$) and the representation of a fourth-order differential expression in the proposed form, the product rule of differential calculus is fulfilled for a simple contraction of second-order tensors. Furthermore, the chain rule can be used. Note that for the classical representation of a differential expression in the form:

$$\frac{\partial\mathbf{A}^\flat}{\partial\mathbf{B}^\flat} = \frac{\partial A_{ij}}{\partial B_{kl}}\boldsymbol{G}^i \otimes \boldsymbol{G}^j \otimes \boldsymbol{G}_k \otimes \boldsymbol{G}_l = \mathbf{A}^\flat{}_{;\mathbf{B}^\flat},$$

which was often used in the past, the product rule cannot be applied. Thus:

$$\begin{aligned}\Delta(\mathbf{A}^\flat\mathbf{B}^\sharp(\mathbf{X}^\flat)) = \frac{\partial\mathbf{A}^\flat\mathbf{B}^\sharp}{\partial\mathbf{X}^\flat} : \Delta\mathbf{X}^\flat &= (\mathbf{A}^\flat\mathbf{B}^\sharp)_{;\mathbf{X}^\flat} : \Delta\mathbf{X}^\flat \\ &\neq (\mathbf{A}^\flat{}_{;\mathbf{X}^\flat}\mathbf{B}^\sharp + \mathbf{A}^\flat\mathbf{B}^\sharp{}_{;\mathbf{X}^\flat}) : \Delta\mathbf{X}^\flat.\end{aligned}$$

Note that for traces of second-order tensors we have to use the contraction rule $(\circ\bullet)$:

$$\frac{\partial \mathrm{tr}(\mathbf{A}^\flat \mathbf{B}^\sharp)}{\partial \mathbf{X}^\flat} = (\mathbf{A}^\flat,_{\mathbf{X}^\flat} \mathbf{B}^\sharp + \mathbf{A}^\flat \mathbf{B}^\sharp,_{\mathbf{X}^\flat})\circ\bullet\mathbf{I}.$$

The contraction rules $(\bullet\circ)$ and $(\circ\bullet)$ can be used to represent the same contraction in absolute notation in different form:

$$\mathbb{E}^{\flat\sharp}\bullet\circ\mathbf{I} = \mathbf{I}\circ\bullet\mathbb{E}^{\flat\sharp},$$

or

$$\frac{\partial \mathrm{tr}(\mathbf{A}^\flat \mathbf{B}^\sharp)}{\partial \mathbf{X}^\flat} = \mathbf{I}\bullet\circ(\mathbf{A}^\flat,_{\mathbf{X}^\flat} \mathbf{B}^\sharp + \mathbf{A}^\flat \mathbf{B}^\sharp,_{\mathbf{X}^\flat}).$$

It is of importance to consider the contraction of tensors which is already existing in a given tensor equation in correct form by using either $(\bullet\circ)$ or $(\circ\bullet)$!

Transformation between the new and old convention:

To transform between $\mathbf{A}^\flat,_{\mathbf{B}^\sharp}$ and $\mathbf{A}^\flat;_{\mathbf{B}^\sharp}$ we can use the following basis rearrangement operations:

$$\begin{array}{lll} (\mathbf{A}\otimes\mathbf{B})^L = \mathbf{A}\times\mathbf{B}\ , & (\mathbf{A}\times\mathbf{B})^L = \mathbf{A}\boxtimes\mathbf{B}^*\ , & (\mathbf{A}\boxtimes\mathbf{B})^L = \mathbf{A}\otimes\mathbf{B}^*\ , \\ (\mathbf{A}\otimes\mathbf{B})^R = \mathbf{A}\boxtimes\mathbf{B}^*\ , & (\mathbf{A}\times\mathbf{B})^R = \mathbf{A}\otimes\mathbf{B}\ , & (\mathbf{A}\boxtimes\mathbf{B})^R = \mathbf{A}\times\mathbf{B}^*\ . \end{array}$$

such that:

$$\mathbf{A}^\flat,_{\mathbf{B}^\sharp} = (\mathbf{A}^\flat;_{\mathbf{B}^\sharp})^L \quad \text{and} \quad \mathbf{A}^\flat;_{\mathbf{B}^\sharp} = (\mathbf{A}^\flat,_{\mathbf{B}^\sharp})^R.$$

Also, applying $(\cdot)^R$ and $(\cdot)^L$, the sequence of tensors is changed in a double contraction:

$$\begin{array}{ll} \mathbb{E}:\mathbf{C} = \mathbb{E}^L\bullet\circ\mathbf{C} = \mathbf{C}\circ\bullet\mathbb{E}^L\ , & \mathbb{E}\bullet\circ\mathbf{C} = \mathbf{C}\circ\bullet\mathbb{E} = \mathbb{E}^R:\mathbf{C}\ , \\ (\mathbb{D}:\mathbb{E})^L = \mathbb{D}^L\bullet\circ\mathbb{E}^L = \mathbb{E}^L\circ\bullet\mathbb{D}^L\ , & (\mathbb{D}\bullet\circ\mathbb{E})^R = (\mathbb{E}\circ\bullet\mathbb{D})^R = \mathbb{D}^R:\mathbb{E}^R\ . \end{array}$$

3 Differential geometrical relationships

Material time derivative of a tensor of first and second order:

$$\dot{\boldsymbol{A}}^\sharp = \dot{A}^i \boldsymbol{G}_i + A^i \dot{\boldsymbol{G}}_i,$$

$$\dot{\mathbf{A}}^\sharp = \dot{A}^{ij} \boldsymbol{G}_i \otimes \boldsymbol{G}_j + A^{ij} \dot{\boldsymbol{G}}_i \otimes \boldsymbol{G}_j + A^{ij} \boldsymbol{G}_i \otimes \dot{\boldsymbol{G}}_j.$$

Push-forward and pull-back-relationships:

$$\begin{aligned}
\mathbf{F}_\triangleright(\boldsymbol{A}^\flat) &= \mathbf{F}^{-*}\boldsymbol{A}^\flat, & \mathbf{F}^\triangleleft(\boldsymbol{a}^\flat) &= \mathbf{F}^{*}\boldsymbol{a}^\flat,\\
\mathbf{F}_\triangleright(\boldsymbol{A}^\sharp) &= \mathbf{F}\boldsymbol{A}^\sharp, & \mathbf{F}^\triangleleft(\boldsymbol{a}^\sharp) &= \mathbf{F}^{-1}\boldsymbol{a}^\sharp,
\end{aligned}$$

$$\begin{aligned}
\mathbf{F}_\triangleright(\mathbf{A}^\flat) &= \mathbf{F}^{-*}\mathbf{A}^\flat\mathbf{F}^{-1}, & \mathbf{F}^\triangleleft(\mathbf{a}^\flat) &= \mathbf{F}^{*}\mathbf{a}^\flat\mathbf{F},\\
\mathbf{F}_\triangleright(\mathbf{A}^\sharp) &= \mathbf{F}\mathbf{A}^\sharp\mathbf{F}^{*}, & \mathbf{F}^\triangleleft(\mathbf{a}^\sharp) &= \mathbf{F}^{-1}\mathbf{a}^\sharp\mathbf{F}^{-*},\\
\mathbf{F}_\triangleright(\mathbf{A}^\backslash) &= \mathbf{F}\mathbf{A}^\backslash\mathbf{F}^{-1}, & \mathbf{F}^\triangleleft(\mathbf{a}^\backslash) &= \mathbf{F}^{-1}\mathbf{a}^\backslash\mathbf{F},\\
\mathbf{F}_\triangleright(\mathbf{A}^/) &= \mathbf{F}^{-*}\mathbf{A}^/\mathbf{F}^{*}, & \mathbf{F}^\triangleleft(\mathbf{a}^/) &= \mathbf{F}^{*}\mathbf{a}^/\mathbf{F}^{-*},
\end{aligned}$$

$$\begin{aligned}
\mathbf{F}_\triangleright(\mathbb{C}^{\flat\flat}) &= (\mathbf{F}^{-*}\otimes\mathbf{F}^{-1})\bullet\!\circ\,\mathbb{C}^{\flat\flat}\bullet\!\circ(\mathbf{F}^{-1}\otimes\mathbf{F}^{-*}),\\
\mathbf{F}^\triangleleft(\mathbb{c}^{\flat\flat}) &= (\mathbf{F}^{*}\otimes\mathbf{F})\bullet\!\circ\,\mathbb{c}^{\flat\flat}\bullet\!\circ(\mathbf{F}\otimes\mathbf{F}^{*}),\\
\mathbf{F}_\triangleright(\mathbb{C}^{\sharp\sharp}) &= (\mathbf{F}\otimes\mathbf{F}^{*})\bullet\!\circ\,\mathbb{C}^{\sharp\sharp}\bullet\!\circ(\mathbf{F}^{*}\otimes\mathbf{F}),\\
\mathbf{F}^\triangleleft(\mathbb{c}^{\sharp\sharp}) &= (\mathbf{F}^{-1}\otimes\mathbf{F}^{-*})\bullet\!\circ\,\mathbb{c}^{\sharp\sharp}\bullet\!\circ(\mathbf{F}^{-*}\otimes\mathbf{F}^{-1}).
\end{aligned}$$

Remark: *Note that* $\mathbf{F}$ *must be an invertible second-order tensor mapping vectors onto vectors i.e.* $\mathbf{F}^\backslash$.

Covariance of tensor functions:

A scalar-valued (second-order-valued, fourth-order-valued) tensor function is covariant if the following equations are satisfied:

$$\begin{aligned}
F(\mathbb{B},\mathbf{B},\mathrm{B}) &= F(\mathbf{A}_\triangleright(\mathbb{B}),\mathbf{A}_\triangleright(\mathbf{B}),\mathrm{B})\ ,\\
&= F(\mathbf{A}^\triangleleft(\mathbb{B}),\mathbf{A}^\triangleleft(\mathbf{B}),\mathrm{B})\ ,\\
\mathbf{A}_\triangleright(\mathbf{F}(\mathbb{B},\mathbf{B},\mathrm{B})) &= \mathbf{F}(\mathbf{A}_\triangleright(\mathbb{B}),\mathbf{A}_\triangleright(\mathbf{B}),\mathrm{B})\ ,\\
\mathbf{A}^\triangleleft(\mathbf{F}(\mathbb{B},\mathbf{B},\mathrm{B})) &= \mathbf{F}(\mathbf{A}^\triangleleft(\mathbb{B}),\mathbf{A}^\triangleleft(\mathbf{B}),\mathrm{B})\ ,\\
\mathbf{A}_\triangleright(\mathbb{F}(\mathbb{B},\mathbf{B},\mathrm{B})) &= \mathbb{F}(\mathbf{A}_\triangleright(\mathbb{B}),\mathbf{A}_\triangleright(\mathbf{B}),\mathrm{B})\ ,\\
\mathbf{A}^\triangleleft(\mathbb{F}(\mathbb{B},\mathbf{B},\mathrm{B})) &= \mathbb{F}(\mathbf{A}^\triangleleft(\mathbb{B}),\mathbf{A}^\triangleleft(\mathbf{B}),\mathrm{B})\ .
\end{aligned}$$

Remark:*Covariance of a tensor function is satisfied if it is constructed using the representation theorem for isotropic tensor functions. Any tensorial invariant (usually some linear combinations of traces) has to be computed in* *mixed-variant form considering certain metric tensors. Normally, we distinguish the principle of material and spatial covariance depending on which manifold the transformed tensors belong to. However, we can speak simply of the principle of covariance whereby the general case is considered where any tensor function may be composed of tensors belonging to different manifolds.*

<u>The Lie-derivative (for convective coordinates ($\boldsymbol{g}_i = \mathbf{F}\boldsymbol{G}_i$):</u>

Definition: $\mathrm{L}_{\mathbf{F}}(\cdot) = \mathbf{F}_{\triangleright}(\dot{\overline{\mathbf{F}^{\triangleleft}(\cdot)}})$.

$$
\begin{aligned}
\mathrm{L}_{\mathbf{F}}(\boldsymbol{a}^{\flat}) &= \mathbf{F}_{\triangleright}(\dot{\overline{\mathbf{F}^{\triangleleft}(\boldsymbol{a}^{\flat})}}) = \mathbf{F}^{-*}(\dot{\overline{\mathbf{F}^{*}\boldsymbol{a}^{\flat}}}) \\
&= \mathbf{F}^{-*}(\dot{\mathbf{F}}^{*}\boldsymbol{a}^{\flat} + \mathbf{F}^{*}\dot{\boldsymbol{a}}^{\flat}) \\
&= \dot{\boldsymbol{a}}^{\flat} + \mathbf{l}^{*}\boldsymbol{a}^{\flat} = \dot{a}_i\,\boldsymbol{g}^i, \\
\mathrm{L}_{\mathbf{F}}(\boldsymbol{a}^{\sharp}) &= \mathbf{F}_{\triangleright}(\dot{\overline{\mathbf{F}^{\triangleleft}(\boldsymbol{a}^{\sharp})}}) = \mathbf{F}(\dot{\overline{\mathbf{F}^{-1}\boldsymbol{a}^{\sharp}}}) \\
&= \mathbf{F}(\dot{\mathbf{F}}^{-1}\boldsymbol{a}^{\sharp} + \mathbf{F}^{-1}\dot{\boldsymbol{a}}^{\sharp}) \\
&= \dot{\boldsymbol{a}}^{\sharp} - \mathbf{l}\boldsymbol{a}^{\sharp} = \dot{a}^i\,\boldsymbol{g}_i.
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{L}_{\mathbf{F}}(\mathbf{a}^{\flat}) &= \mathbf{F}_{\triangleright}(\dot{\overline{\mathbf{F}^{\triangleleft}(\mathbf{a}^{\flat})}}) = \mathbf{F}^{-*}(\dot{\overline{\mathbf{F}^{*}\mathbf{a}^{\flat}\mathbf{F}}})\mathbf{F}^{-1} \\
&= \mathbf{F}^{-*}(\dot{\mathbf{F}}^{*}\mathbf{a}^{\flat}\mathbf{F} + \mathbf{F}^{*}\dot{\mathbf{a}}^{\flat}\mathbf{F} + \mathbf{F}^{*}\mathbf{a}^{\flat}\dot{\mathbf{F}})\mathbf{F}^{-1} \\
&= \dot{\mathbf{a}}^{\flat} + \mathbf{l}^{*}\mathbf{a}^{\flat} + \mathbf{a}^{\flat}\mathbf{l} = \dot{a}_{ij}\,\boldsymbol{g}^i \otimes \boldsymbol{g}^j, \\
\mathrm{L}_{\mathbf{F}}(\mathbf{a}^{\sharp}) &= \mathbf{F}_{\triangleright}(\dot{\overline{\mathbf{F}^{\triangleleft}(\mathbf{a}^{\sharp})}}) = \mathbf{F}(\dot{\overline{\mathbf{F}^{-1}\mathbf{a}^{\sharp}\mathbf{F}^{-*}}})\mathbf{F}^{*} \\
&= \mathbf{F}(\dot{\mathbf{F}}^{-1}\mathbf{a}^{\sharp}\mathbf{F}^{-*} + \mathbf{F}^{-1}\dot{\mathbf{a}}^{\sharp}\mathbf{F}^{-*} + \mathbf{F}^{-1}\mathbf{a}^{\sharp}\dot{\mathbf{F}}^{-*})\mathbf{F}^{*} \\
&= \dot{\mathbf{a}}^{\sharp} - \mathbf{l}\mathbf{a}^{\sharp} - \mathbf{a}^{\sharp}\mathbf{l}^{*} = \dot{a}^{ij}\,\boldsymbol{g}_i \otimes \boldsymbol{g}_j, \\
\mathrm{L}_{\mathbf{F}}(\mathbf{a}^{\backslash}) &= \mathbf{F}_{\triangleright}(\dot{\overline{\mathbf{F}^{\triangleleft}(\mathbf{a}^{\backslash})}}) = \mathbf{F}(\dot{\overline{\mathbf{F}^{-1}\mathbf{a}^{\backslash}\mathbf{F}}})\mathbf{F}^{-1} \\
&= \mathbf{F}(\dot{\mathbf{F}}^{-1}\mathbf{a}^{\backslash}\mathbf{F} + \mathbf{F}^{-1}\dot{\mathbf{a}}^{\backslash}\mathbf{F} + \mathbf{F}^{-1}\mathbf{a}^{\backslash}\dot{\mathbf{F}})\mathbf{F}^{-1} \\
&= \dot{\mathbf{a}}^{\backslash} - \mathbf{l}\mathbf{a}^{\backslash} + \mathbf{a}^{\backslash}\mathbf{l} = \dot{a}^{i}_{.j}\,\boldsymbol{g}_i \otimes \boldsymbol{g}^j, \\
\mathrm{L}_{\mathbf{F}}(\mathbf{a}^{/}) &= \mathbf{F}_{\triangleright}(\dot{\overline{\mathbf{F}^{\triangleleft}(\mathbf{a}^{/})}}) = \mathbf{F}^{-*}(\dot{\overline{\mathbf{F}^{*}\mathbf{a}^{/}\mathbf{F}^{-*}}})\mathbf{F}^{*} \\
&= \mathbf{F}^{-*}(\dot{\mathbf{F}}^{*}\mathbf{a}^{/}\mathbf{F}^{-*} + \mathbf{F}^{*}\dot{\mathbf{a}}^{/}\mathbf{F}^{-*} + \mathbf{F}^{*}\mathbf{a}^{/}\dot{\mathbf{F}}^{-*})\mathbf{F}^{*} \\
&= \dot{\mathbf{a}}^{/} + \mathbf{l}^{*}\mathbf{a}^{/} - \mathbf{a}^{/}\mathbf{l}^{*} = \dot{a}_{i.}^{\ j}\,\boldsymbol{g}^i \otimes \boldsymbol{g}_j.
\end{aligned}
$$

A time derivative is not unique:

We are able to use any time derivative (material, Lie, co-rotated) for the time differentiation of covariant tensor functions. Thus, the choice of a time derivative is not unique and can only be motivated on the grounds of special constitutive assumptions. Thus, we find:

$$
\begin{aligned}
\overline{F(\mathbf{a}^\flat,\mathbf{b}^\sharp)}^{\cdot} &= \frac{\partial F(\mathbf{a}^\flat,\mathbf{b}^\sharp)}{\partial \mathbf{a}^\flat} : \dot{\mathbf{a}}^\flat + \frac{\partial F(\mathbf{a}^\flat,\mathbf{b}^\sharp)}{\partial \mathbf{b}^\sharp} : \dot{\mathbf{b}}^\sharp \\
&= \frac{\partial F(\mathbf{c}^{\setminus\triangleleft}(\mathbf{a}^\flat),\mathbf{c}^{\setminus\triangleleft}(\mathbf{b}^\sharp))}{\partial \mathbf{c}^{\setminus\triangleleft}(\mathbf{a}^\flat)} : \overline{\mathbf{c}^{\setminus\triangleleft}(\mathbf{a}^\flat)}^{\cdot} + \frac{\partial F(\mathbf{c}^{\setminus\triangleleft}(\mathbf{a}^\flat),\mathbf{c}^{\setminus\triangleleft}(\mathbf{b}^\sharp))}{\partial \mathbf{c}^{\setminus\triangleleft}(\mathbf{b}^\sharp)} : \overline{\mathbf{c}^{\setminus\triangleleft}(\mathbf{b}^\sharp)}^{\cdot} \\
&= \mathbf{c}^{\setminus\triangleleft}\left(\frac{\partial F(\mathbf{a}^\flat,\mathbf{b}^\sharp)}{\partial \mathbf{a}^\flat}\right) : \overline{\mathbf{c}^{\setminus\triangleleft}(\mathbf{a}^\flat)}^{\cdot} + \mathbf{c}^{\setminus\triangleleft}\left(\frac{\partial F(\mathbf{a}^\flat,\mathbf{b}^\sharp)}{\partial \mathbf{b}^\sharp}\right) : \overline{\mathbf{c}^{\setminus\triangleleft}(\mathbf{b}^\sharp)}^{\cdot} \\
&= \frac{\partial F(\mathbf{a}^\flat,\mathbf{b}^\sharp)}{\partial \mathbf{a}^\flat} : \mathbf{c}^{\setminus}{}_{\triangleright}(\overline{\mathbf{c}^{\setminus\triangleleft}(\mathbf{a}^\flat)}^{\cdot}) + \frac{\partial F(\mathbf{a}^\flat,\mathbf{b}^\sharp)}{\partial \mathbf{b}^\sharp} : \mathbf{c}^{\setminus}{}_{\triangleright}(\overline{\mathbf{c}^{\setminus\triangleleft}(\mathbf{b}^\sharp)}^{\cdot}) \\
&= \frac{\partial F(\mathbf{a}^\flat,\mathbf{b}^\sharp)}{\partial \mathbf{a}^\flat} : \mathrm{L}_{\mathbf{c}^\setminus}(\mathbf{a}^\flat) + \frac{\partial F(\mathbf{a}^\flat,\mathbf{b}^\sharp)}{\partial \mathbf{b}^\sharp} : \mathrm{L}_{\mathbf{c}^\setminus}(\mathbf{b}^\sharp) = \mathrm{L}_{\mathbf{c}^\setminus}(F)\ .
\end{aligned}
$$

or

$$
\begin{aligned}
\mathrm{L}_{\mathbf{c}_1^\setminus}(\mathbf{F}(\mathbf{c}_2^{\setminus\triangleleft}(\mathbf{a}^\flat),\mathbf{c}_2^{\setminus\triangleleft}(\mathbf{b}^\sharp))) &= \frac{\partial \mathbf{F}(\mathbf{c}_2^{\setminus\triangleleft}(\mathbf{a}^\flat),\mathbf{c}_2^{\setminus\triangleleft}(\mathbf{b}^\sharp))}{\partial \mathbf{c}_2^{\setminus\triangleleft}(\mathbf{a}^\flat)} {\bullet\circ} \mathrm{L}_{\mathbf{c}_1^\setminus}(\mathbf{c}_2^{\setminus\triangleleft}(\mathbf{a}^\flat)) + \cdots \\
&= \frac{\partial \mathbf{F}(\mathbf{c}_2^{\setminus\triangleleft}(\mathbf{a}^\flat),\mathbf{c}_2^{\setminus\triangleleft}(\mathbf{b}^\sharp))}{\mathbf{c}^{\setminus}_{2\triangleright}(\partial \mathbf{c}_2^{\setminus\triangleleft}(\mathbf{a}^\flat))} {\bullet\circ} \mathbf{c}^{\setminus}_{2\triangleright}(\mathrm{L}_{\mathbf{c}_1^\setminus}(\mathbf{c}_2^{\setminus\triangleleft}(\mathbf{a}^\flat))) + \cdots \\
&= \frac{\partial \mathbf{c}_2^{\setminus\triangleleft}(\mathbf{F}(\mathbf{a}^\flat,\mathbf{b}^\sharp))}{\partial \mathbf{a}^\flat} {\bullet\circ} \mathrm{L}_{\mathbf{c}_2^\setminus \mathbf{c}_1^\setminus}(\mathbf{a}^\flat) + \cdots \\
&= \frac{\mathbf{c}_2^{\setminus\triangleleft}(\partial \mathbf{F}(\mathbf{a}^\flat,\mathbf{b}^\sharp))}{\partial \mathbf{a}^\flat} {\bullet\circ} \mathrm{L}_{\mathbf{c}_2^\setminus \mathbf{c}_1^\setminus}(\mathbf{a}^\flat) + \cdots \\
&= \mathbf{c}_2^{\setminus\triangleleft}(\mathrm{L}_{\mathbf{c}_2^\setminus \mathbf{c}_1^\setminus}(\mathbf{F}(\mathbf{a}^\flat,\mathbf{b}^\sharp)))\ .
\end{aligned}
$$

Remark: *Note in particular that* $\dfrac{\partial(\cdot)(\mathbf{c}^{\setminus\triangleleft}(\mathbf{a}^\flat),\mathbf{c}^{\setminus\triangleleft}(\mathbf{b}^\sharp))}{\partial \mathbf{c}^{\setminus\triangleleft}(\mathbf{a}^\flat)} = \mathbf{c}^{\setminus\triangleleft}\left(\dfrac{\partial(\cdot)(\mathbf{a}^\flat,\mathbf{b}^\sharp)}{\partial \mathbf{a}^\flat}\right).$

The following couplings hold for a scalar-valued function $f(\mathbf{a}^\flat_j, \mathbf{b}^\sharp_i, \mathbf{c}^\backslash_k, \mathbf{d}^/_l)$:

$$-\sum_{j=1}^{m1} (\mathbf{a}^\flat)^*_j \frac{\partial F}{\partial(\mathbf{a}^\flat)_j} + \sum_{i=1}^{m2} \frac{\partial F}{\partial(\mathbf{b}^\sharp)_i}(\mathbf{b}^\sharp)^*_i + \sum_{k=1}^{m3} \left(\frac{\partial F}{\partial(\mathbf{c}^\backslash)_k}(\mathbf{c}^\backslash)^*_k - (\mathbf{c}^\backslash)^*_k \frac{\partial F}{\partial(\mathbf{c}^\backslash)_k} \right) = \mathbf{0},$$

$j \in [1, \ldots, m1],\ i \in [1, \ldots, m2], k \in [1, \ldots, m3],$

and

$$-\sum_{j=1}^{m1} \frac{\partial F}{\partial(\mathbf{a}^\flat)_j}(\mathbf{a}^\flat)^*_j + \sum_{i=1}^{m2} (\mathbf{b}^\sharp)^*_i \frac{\partial F}{\partial(\mathbf{b}^\sharp)_i} + \sum_{l=1}^{m4} \left((\mathbf{d}^/)^*_l \frac{\partial F}{\partial(\mathbf{d}^/)_l} - \frac{\partial F}{\partial(\mathbf{d}^/)_l}(\mathbf{d}^/)^*_l \right) = \mathbf{0},$$

$j \in [1, \ldots, m1],\ i \in [1, \ldots, m2], l \in [1, \ldots, m4].$

Remark: *A valid function F would be e.g.*

$$F = \mathrm{tr}((\mathbf{d}^/)^2 \mathbf{a}^\flat \mathbf{b}^\sharp \mathbf{a}^\flat \mathbf{b}^\sharp)\,\mathrm{tr}((\mathbf{c}^\backslash)^2 \mathbf{b}^\sharp \mathbf{a}^\flat \mathbf{b}^\sharp \mathbf{a}^\flat).$$

For a second-order-valued tensor function only very restricted cases are considered. E.g. we consider the functions:

$\mathbf{F}^\backslash(\mathbf{a}^\flat_j, \mathbf{b}^\sharp_i)$ and $\mathbf{F}^/(\mathbf{a}^\flat_j, \mathbf{b}^\sharp_i)$.

Then the following couplings hold:

$$-\sum_{j=1}^{m1} \frac{\partial \mathbf{F}^\backslash}{\partial(\mathbf{a}^\flat)_j} \bullet\!\circ (\mathbf{i}^* \otimes (\mathbf{a}^\flat)_j) + \sum_{i=1}^{m2} \frac{\partial \mathbf{F}^\backslash}{\partial(\mathbf{b}^\sharp)_i} \bullet\!\circ ((\mathbf{b}^\sharp)_i \otimes \mathbf{i}^*) = \mathbf{0},$$

$$-\sum_{j=1}^{m1} \frac{\partial \mathbf{F}^/}{\partial(\mathbf{a}^\flat)_j} \bullet\!\circ ((\mathbf{a}^\flat)_j \otimes \mathbf{i}) + \sum_{i=1}^{m2} \frac{\partial \mathbf{F}^/}{\partial(\mathbf{b}^\sharp)_i} \bullet\!\circ (\mathbf{i} \otimes (\mathbf{b}^\sharp)_i) = \mathbf{0},$$

$j \in [1, \ldots, m1],\ i \in [1, \ldots, m2].$

For the second-order derivative of the above introduced scalar-valued function similar couplings hold! Thus, we find:

$$
\begin{aligned}
&\sum_{j=1}^{m1}\sum_{i=1}^{m2}\left(\mathbb{A}_{2i}^{T}\bullet\circ\frac{\partial^2 F}{\partial(\mathbf{b}^\sharp)_i\partial(\mathbf{a}^\flat)_j}\bullet\circ\mathbb{A}_{1j}+\ \mathbb{A}_{1j}^{T}\bullet\circ\frac{\partial^2 F}{\partial(\mathbf{a}^\flat)_j\partial(\mathbf{b}^\sharp)_i}\bullet\circ\mathbb{A}_{2i}\right)\\
&+\sum_{j=1}^{m1}\sum_{k=1}^{m3}\left(\mathbb{A}_{3k}^{T}\bullet\circ\frac{\partial^2 F}{\partial(\mathbf{c}^\natural)_k\partial(\mathbf{a}^\flat)_j}\bullet\circ\mathbb{A}_{1j}+\ \mathbb{A}_{1j}^{T}\bullet\circ\frac{\partial^2 F}{\partial(\mathbf{a}^\flat)_j\partial(\mathbf{c}^\natural)_k}\bullet\circ\mathbb{A}_{3k}\right)\\
&+\sum_{i=1}^{m2}\sum_{k=1}^{m3}\left(\mathbb{A}_{3k}^{T}\bullet\circ\frac{\partial^2 F}{\partial(\mathbf{c}^\natural)_k\partial(\mathbf{b}^\sharp)_i}\bullet\circ\mathbb{A}_{2i}+\ \mathbb{A}_{2i}^{T}\bullet\circ\frac{\partial^2 F}{\partial(\mathbf{b}^\sharp)_i\partial(\mathbf{c}^\natural)_k}\bullet\circ\mathbb{A}_{3k}\right)\\
&+\sum_{i,k=1}^{m1}\ \mathbb{A}_{1i}^{T}\bullet\circ\frac{\partial^2 F}{\partial(\mathbf{a}^\flat)_i\partial(\mathbf{a}^\flat)_k}\bullet\circ\mathbb{A}_{1k}+\sum_{j,l=1}^{m2}\ \mathbb{A}_{2j}^{T}\bullet\circ\frac{\partial^2 F}{\partial(\mathbf{b}^\sharp)_j\partial(\mathbf{b}^\sharp)_l}\bullet\circ\mathbb{A}_{2l}\\
&+\sum_{i,k=1}^{m3}\ \mathbb{A}_{3i}^{T}\bullet\circ\frac{\partial^2 F}{\partial(\mathbf{c}^\natural)_i\partial(\mathbf{c}^\natural)_k}\bullet\circ\mathbb{A}_{3k}+\mathbb{G}=\mathbf{0},
\end{aligned}
$$

with

$$
\mathbb{A}_{1j}=-((\mathbf{a}^\flat)_j\otimes\mathbf{i}),\quad \mathbb{A}_{2i}=(\mathbf{i}\otimes(\mathbf{b}^\sharp)_i),\quad \mathbb{A}_{3k}=-((\mathbf{c}^\natural)_k\otimes\mathbf{i})+(\mathbf{i}\otimes(\mathbf{c}^\natural)_k),
$$

$$
\begin{aligned}
\mathbb{G}\ =\ &\textstyle\sum_{j=1}^{m1}\mathbf{i}^*\boxtimes(\mathbf{a}^\flat)_j^*\frac{\partial F}{\partial(\mathbf{a}^\flat)_j}+\sum_{j=1}^{m1}(\mathbf{a}^\flat)_j^*\frac{\partial F}{\partial(\mathbf{a}^\flat)_j}\boxtimes\mathbf{i}^*+\sum_{k=1}^{m3}\mathbf{i}^*\boxtimes(\mathbf{c}^\natural)_k^*\frac{\partial F}{\partial(\mathbf{c}^\natural)_k}\\
&\textstyle+\sum_{k=1}^{m3}(\mathbf{c}^\natural)_k^*\frac{\partial F}{\partial(\mathbf{c}^\natural)_k}\boxtimes\mathbf{i}^*-\sum_{k=1}^{m3}(\mathbf{c}^\natural)_k^*\boxtimes\frac{\partial F}{\partial(\mathbf{c}^\natural)_k}-\sum_{k=1}^{m3}\frac{\partial F}{\partial(\mathbf{c}^\natural)_k}\boxtimes(\mathbf{c}^\natural)_k^*,
\end{aligned}
$$

and:

$$
\begin{aligned}
&\sum_{j=1}^{m1}\sum_{i=1}^{m2}\left(\mathbb{A}_{2i}^{T}\bullet\circ\frac{\partial^2 F}{\partial(\mathbf{b}^\sharp)_i\partial(\mathbf{a}^\flat)_j}\bullet\circ\mathbb{A}_{1j}+\ \mathbb{A}_{1j}^{T}\bullet\circ\frac{\partial^2 F}{\partial(\mathbf{a}^\flat)_j\partial(\mathbf{b}^\sharp)_i}\bullet\circ\mathbb{A}_{2i}\right)\\
&+\sum_{j=1}^{m1}\sum_{l=1}^{m4}\left(\mathbb{A}_{4l}^{T}\bullet\circ\frac{\partial^2 F}{\partial(\mathbf{d}^/)_l\partial(\mathbf{a}^\flat)_j}\bullet\circ\mathbb{A}_{1j}+\ \mathbb{A}_{1j}^{T}\bullet\circ\frac{\partial^2 F}{\partial(\mathbf{a}^\flat)_j\partial(\mathbf{d}^/)_l}\bullet\circ\mathbb{A}_{4l}\right)\\
&+\sum_{i=1}^{m2}\sum_{l=1}^{m4}\left(\mathbb{A}_{4l}^{T}\bullet\circ\frac{\partial^2 F}{\partial(\mathbf{d}^/)_l\partial(\mathbf{b}^\sharp)_i}\bullet\circ\mathbb{A}_{2i}+\ \mathbb{A}_{2i}^{T}\bullet\circ\frac{\partial^2 F}{\partial(\mathbf{b}^\sharp)_i\partial(\mathbf{d}^/)_l}\bullet\circ\mathbb{A}_{4l}\right)\\
&+\sum_{i,k=1}^{m1}\ \mathbb{A}_{1i}^{T}\bullet\circ\frac{\partial^2 F}{\partial(\mathbf{a}^\flat)_i\partial(\mathbf{a}^\flat)_k}\bullet\circ\mathbb{A}_{1k}+\sum_{j,l=1}^{m2}\ \mathbb{A}_{2j}^{T}\bullet\circ\frac{\partial^2 F}{\partial(\mathbf{b}^\sharp)_j\partial(\mathbf{b}^\sharp)_l}\bullet\circ\mathbb{A}_{2l}\\
&+\sum_{j,l=1}^{m4}\ \mathbb{A}_{4j}^{T}\bullet\circ\frac{\partial^2 W}{\partial(\mathbf{d}^/)_j\partial(\mathbf{d}^/)_l}\bullet\circ\mathbb{A}_{4l}+\mathbb{G}=\mathbf{0},
\end{aligned}
$$

with

$$
\mathbb{A}_{1j}=-(\mathbf{i}^*\otimes(\mathbf{a}^\flat)_j),\quad \mathbb{A}_{2i}=((\mathbf{b}^\sharp)_i\otimes\mathbf{i}^*),\quad \mathbb{A}_{4l}=-(\mathbf{i}^*\otimes(\mathbf{d}^/)_l)+((\mathbf{d}^/)_l\otimes\mathbf{i}^*),
$$

$$
\begin{aligned}
\mathbb{G}\ =\ &\textstyle\sum_{j=1}^{m1}\mathbf{i}\boxtimes\frac{\partial F}{\partial(\mathbf{a}^\flat)_j}(\mathbf{a}^\flat)_j^*+\sum_{j=1}^{m1}\frac{\partial F}{\partial(\mathbf{a}^\flat)_j}(\mathbf{a}^\flat)_j^*\boxtimes\mathbf{i}+\sum_{l=1}^{m4}\frac{\partial F}{\partial(\mathbf{d}^/)_l}(\mathbf{d}^/)_l^*\boxtimes\mathbf{i}\\
&\textstyle+\sum_{l=1}^{m4}\mathbf{i}\boxtimes\frac{\partial F}{\partial(\mathbf{d}^/)_l}(\mathbf{d}^/)_l^*-\sum_{l=1}^{m4}\frac{\partial F}{\partial(\mathbf{d}^/)_l}\boxtimes(\mathbf{d}^/)_l^*-\sum_{l=1}^{m4}(\mathbf{d}^/)_l^*\boxtimes\frac{\partial F}{\partial(\mathbf{d}^/)_l},
\end{aligned}
$$

$j\in[1,\ldots,m1],\ i\in[1,\ldots,m2], k\in[1,\ldots,m3], l\in[1,\ldots,m4].$

The Lie-variation:

$$\delta_{\mathbf{F}} f(\mathbf{X}) = \mathbf{F}_{\triangleright}(\delta(\mathbf{F}^{\triangleleft}(f(\mathbf{X})))) = \mathbf{F}_{\triangleright}(\frac{d}{d\epsilon}\left(\mathbf{F}^{\triangleleft}(f(\mathbf{X}+\epsilon\,\delta\mathbf{X}))\right)\Big|_{\epsilon=0})$$

including the typical variation as special case:

$$\delta f(\mathbf{X}) = \frac{d}{d\epsilon} f(\mathbf{X}+\epsilon\,\delta\mathbf{X})\Big|_{\epsilon=0} = \frac{\partial f}{\partial \mathbf{X}}\Big|_{\delta\mathbf{X}=\mathbf{0}} \bullet\circ \delta\mathbf{X}.$$

Analogously, the variation of a second-order tensor function can be found:

$$\delta\mathbf{A}(\mathbf{X}) = \frac{d}{d\epsilon}\mathbf{A}(\mathbf{X}+\delta\mathbf{X})\Big|_{\epsilon=0} = \frac{\partial \mathbf{A}}{\partial \mathbf{X}}\Big|_{\delta\mathbf{X}=\mathbf{0}} \bullet\circ \delta\mathbf{X},$$

or its general Lie-variation:

$$\delta_{\mathbf{F}}\mathbf{A}(\mathbf{X}) = \mathbf{F}_{\triangleright}(\frac{d}{d\epsilon}\left(\mathbf{F}^{\triangleleft}(\mathbf{A}(\mathbf{X}+\delta\mathbf{X}))\right)\Big|_{\epsilon=0}).$$

The linearization of a second-order tensor function can be computed as:

$$\mathrm{Lin}(\mathbf{A}(\mathbf{X},\Delta\mathbf{X})) = \mathbf{A}(\mathbf{X}) + \Delta\mathbf{A}(\mathbf{X},\Delta\mathbf{X}) = \mathbf{A}(\mathbf{X}) + \frac{\partial \mathbf{A}}{\partial \mathbf{X}}\Big|_{\Delta\mathbf{X}=0} \bullet\circ \Delta\mathbf{X}.$$

The gradient and divergence of a first and second-order tensor:

$$\mathrm{Grad}(\boldsymbol{A}^{\sharp}) = \frac{\partial \boldsymbol{A}^{\sharp}}{\partial \boldsymbol{X}} = \frac{\partial \boldsymbol{A}^{\sharp}}{\partial \theta^{i}} \otimes \boldsymbol{G}^{i},$$

$$\mathrm{Grad}(\mathbf{A}^{\sharp}) = \frac{\partial \mathbf{A}^{\sharp}}{\partial \boldsymbol{X}} = \frac{\partial \mathbf{A}^{\sharp}}{\partial \theta^{i}} \otimes \boldsymbol{G}^{i},$$

$$\mathrm{Div}(\boldsymbol{A}^{\sharp}) = \mathrm{Grad}(\boldsymbol{A}^{\sharp}) : \mathbf{I}^{*} = \frac{\partial A^{i}}{\partial \theta^{i}},$$

$$\mathrm{Div}(\mathbf{A}^{\sharp}) = \mathrm{Grad}(\mathbf{A}^{\sharp}) : \mathbf{I}^{*} = \frac{\partial A^{ik}}{\partial \theta^{k}} \boldsymbol{G}_{i}.$$

In particular, we have:

$$\mathrm{Div}(\mathbf{A}^{\sharp}\boldsymbol{A}^{\flat}) = \mathrm{Grad}(\mathbf{A}^{\sharp}\boldsymbol{A}^{\flat}) : \mathbf{I}^{*} = \boldsymbol{A}^{\flat}\mathrm{Div}((\mathbf{A}^{\sharp})^{*}) + (\mathbf{A}^{\sharp})^{*} : \mathrm{Grad}(\boldsymbol{A}^{\flat}),$$

$$\mathrm{Div}(\boldsymbol{A}^{\flat}\mathbf{A}^{\sharp}) = \mathrm{Grad}(\boldsymbol{A}^{\flat}\mathbf{A}^{\sharp}) : \mathbf{I}^{*} = \boldsymbol{A}^{\flat}\mathrm{Div}(\mathbf{A}^{\sharp}) + \mathbf{A}^{\sharp} : \mathrm{Grad}(\boldsymbol{A}^{\flat}).$$

4 Application of the tensor differentiation rules

Here, we use an invariant representation of tensors. In fact, these tensors could have arbitrary component variance. Although the actual kind of component variance may be important for the implementation process, the use of invariant relations merits consideration for their simple expressions (e.g. $\mathbf{I}^*$ is simply written as $\mathbf{I}$ and no symbols $(\cdot)^\flat$, $(\cdot)^\sharp$, $(\cdot)^\backslash$ or $(\cdot)^/$).

Differentiation of $F(\mathbf{A}) = \mathrm{tr}(\mathbf{A}^3)$:

$$\begin{aligned} F(\mathbf{A})_{,\mathbf{A}} &= \mathbf{I}_{\bullet\circ}((\mathbf{I}\otimes\mathbf{I})\mathbf{A}^2 + \mathbf{A}(\mathbf{I}\otimes\mathbf{I})\mathbf{A} + \mathbf{A}^2(\mathbf{I}\otimes\mathbf{I})) \\ &= \mathbf{I}_{\bullet\circ}(\mathbf{I}\otimes\mathbf{A}^2 + \mathbf{A}\otimes\mathbf{A} + \mathbf{A}^2\otimes\mathbf{I}) \\ &= 3\,(\mathbf{A}^2)^*. \end{aligned}$$

Rule: $\mathrm{tr}(\mathbf{A}^n)_{,\mathbf{A}} = n\,(\mathbf{A}^{n-1})^*$.

Differentiation of $F(\mathbf{A}) = \mathrm{tr}(\mathbf{A}^3)\mathrm{tr}(\mathbf{A}^2)$:

$$F(\mathbf{A})_{,\mathbf{A}} = 3\,(\mathbf{A}^2)^*\,\mathrm{tr}(\mathbf{A}^2) + \mathrm{tr}(\mathbf{A}^3)\,2\,\mathbf{A}^*,$$

$$\begin{aligned} F(\mathbf{A})_{,\mathbf{A}\times\mathbf{A}} = {} & 6((\mathbf{A}^2)^*\times\mathbf{A}^* + \mathbf{A}^*\times(\mathbf{A}^2)^*) \\ & +3\,\mathrm{tr}(\mathbf{A}^2)\,(\mathbf{A}^*\boxtimes\mathbf{I} + \mathbf{I}\boxtimes\mathbf{A}^*) + 2\,\mathrm{tr}(\mathbf{A}^3)\,(\mathbf{I}\boxtimes\mathbf{I}). \end{aligned}$$

Differentiation of $\mathbf{F}(\mathbf{A}) = \mathbf{A}\mathrm{tr}(\mathbf{A}^2) + \mathbf{A}^*\mathrm{tr}(\mathbf{A}^3)$:

$$\mathbf{F}(\mathbf{A})_{,\mathbf{A}} = \mathrm{tr}(\mathbf{A}^2)\mathbf{I}\otimes\mathbf{I} + \mathrm{tr}(\mathbf{A}^3)\,\mathbf{I}\boxtimes\mathbf{I} + 2\,\mathbf{A}\times\mathbf{A}^* + 3\,\mathbf{A}^*\times(\mathbf{A}^2)^*.$$

Differentiation of $\mathbf{F}(\mathbf{A}) = \mathrm{dev}(\mathbf{A})$:

$$\mathbf{F}(\mathbf{A}) = \mathbf{A} - \tfrac{1}{3}\mathrm{tr}(\mathbf{A})\,\mathbf{I},$$

$$\mathbf{F}(\mathbf{A})_{,\mathbf{A}} = \mathbf{I}\otimes\mathbf{I} - \tfrac{1}{3}\,\mathbf{I}\times\mathbf{I}.$$

for a symmetric tensor we have: $\mathbf{F}(\mathbf{A})_{,\mathbf{A}} = \mathbb{S} - \frac{1}{3}\,\mathbf{I}\times\mathbf{I}$.

Differentiation of $\mathbf{F}(\mathbf{A}) = (\mathrm{dev}(\mathbf{A}))^3$:

$$\begin{aligned} \mathbf{F}(\mathbf{A})_{,\mathbf{A}} &= ((\mathrm{dev}(\mathbf{A}))^2\otimes\mathbf{I} + \mathrm{dev}(\mathbf{A})\otimes\mathrm{dev}(\mathbf{A}) + \mathbf{I}\otimes(\mathrm{dev}(\mathbf{A}))^2)_{\bullet\circ}(\mathrm{dev}(\mathbf{A}))_{,\mathbf{A}} \\ &= ((\mathrm{dev}(\mathbf{A}))^2\otimes\mathbf{I} + \mathrm{dev}(\mathbf{A})\otimes\mathrm{dev}(\mathbf{A}) + \mathbf{I}\otimes(\mathrm{dev}(\mathbf{A}))^2)_{\bullet\circ}(\mathbf{I}\otimes\mathbf{I} - \tfrac{1}{3}\,\mathbf{I}\times\mathbf{I}) \\ &= (\mathrm{dev}(\mathbf{A}))^2\otimes\mathbf{I} + \mathrm{dev}(\mathbf{A})\otimes\mathrm{dev}(\mathbf{A}) + \mathbf{I}\otimes(\mathrm{dev}(\mathbf{A}))^2 - (\mathrm{dev}(\mathbf{A}))^2\times\mathbf{I}. \end{aligned}$$

5 Tables

$(\mathbf{A} \otimes \mathbf{B})\,\mathbf{C} = \mathbf{A} \otimes (\mathbf{B}\,\mathbf{C})$	$\mathbf{C}\,(\mathbf{A} \otimes \mathbf{B}) = (\mathbf{C}\,\mathbf{A}) \otimes \mathbf{B}$
$(\mathbf{A} \times \mathbf{B})\,\mathbf{C} = (\mathbf{A}\,\mathbf{C}) \times \mathbf{B}$	$\mathbf{C}\,(\mathbf{A} \times \mathbf{B}) = (\mathbf{C}\,\mathbf{A}) \times \mathbf{B}$
$(\mathbf{A} \boxtimes \mathbf{B})\,\mathbf{C} = \mathbf{A} \boxtimes (\mathbf{B}\,\mathbf{C})$	$\mathbf{C}\,(\mathbf{A} \boxtimes \mathbf{B}) = (\mathbf{C}\,\mathbf{A}) \boxtimes \mathbf{B}$

Table 1: Simple contractions of tensors of second and fourth order.

$(\mathbf{A} \otimes \mathbf{B}) : \mathbf{C} = \mathbf{A}\,(\mathbf{B} : \mathbf{C})$	$\mathbf{C} : (\mathbf{A} \otimes \mathbf{B}) = (\mathbf{C} : \mathbf{A})\,\mathbf{B}$
$(\mathbf{A} \otimes \mathbf{B}) \bullet\circ\, \mathbf{C} = \mathbf{A}\,\mathbf{C}\,\mathbf{B}$	$\mathbf{C} \bullet\circ (\mathbf{A} \otimes \mathbf{B}) = \mathbf{A}^*\,\mathbf{C}\,\mathbf{B}^*$
$(\mathbf{A} \otimes \mathbf{B}) \circ\bullet\, \mathbf{C} = \mathbf{A}^*\,\mathbf{C}\,\mathbf{B}^*$	$\mathbf{C} \circ\bullet (\mathbf{A} \otimes \mathbf{B}) = \mathbf{A}\,\mathbf{C}\,\mathbf{B}$
$(\mathbf{A} \times \mathbf{B}) : \mathbf{C} = \mathbf{A}\,\mathbf{C}^*\,\mathbf{B}^*$	$\mathbf{C} : (\mathbf{A} \times \mathbf{B}) = \mathbf{B}^*\,\mathbf{C}^*\,\mathbf{A}$
$(\mathbf{A} \times \mathbf{B}) \bullet\circ\, \mathbf{C} = (\mathbf{B} : \mathbf{C})\,\mathbf{A}$	$\mathbf{C} \bullet\circ (\mathbf{A} \times \mathbf{B}) = (\mathbf{A} : \mathbf{C})\,\mathbf{B}$
$(\mathbf{A} \times \mathbf{B}) \circ\bullet\, \mathbf{C} = (\mathbf{A} : \mathbf{C})\,\mathbf{B}$	$\mathbf{C} \circ\bullet (\mathbf{A} \times \mathbf{B}) = (\mathbf{C} : \mathbf{B})\,\mathbf{A}$
$(\mathbf{A} \boxtimes \mathbf{B}) : \mathbf{C} = \mathbf{A}\,\mathbf{C}\,\mathbf{B}^*$	$\mathbf{C} : (\mathbf{A} \boxtimes \mathbf{B}) = \mathbf{A}^*\,\mathbf{C}\,\mathbf{B}$
$(\mathbf{A} \boxtimes \mathbf{B}) \bullet\circ\, \mathbf{C} = \mathbf{A}\,\mathbf{C}^*\,\mathbf{B}$	$\mathbf{C} \bullet\circ (\mathbf{A} \boxtimes \mathbf{B}) = \mathbf{B}\,\mathbf{C}^*\,\mathbf{A}$
$(\mathbf{A} \boxtimes \mathbf{B}) \circ\bullet\, \mathbf{C} = \mathbf{B}\,\mathbf{C}^*\,\mathbf{A}$	$\mathbf{C} \circ\bullet (\mathbf{A} \boxtimes \mathbf{B}) = \mathbf{A}\,\mathbf{C}^*\,\mathbf{B}$

Table 2: Double contractions of tensors of second and fourth order.

$(\mathbf{A}\otimes\mathbf{B}):(\mathbf{C}\otimes\mathbf{D}) = (\mathbf{B}:\mathbf{C})\,\mathbf{A}\otimes\mathbf{D}$ $(\mathbf{A}\otimes\mathbf{B}){\bullet}{\circ}(\mathbf{C}\otimes\mathbf{D}) = (\mathbf{A}\,\mathbf{C})\otimes(\mathbf{D}\,\mathbf{B})$ $(\mathbf{A}\otimes\mathbf{B}){\circ}{\bullet}(\mathbf{C}\otimes\mathbf{D}) = (\mathbf{C}\,\mathbf{A})\otimes(\mathbf{B}\,\mathbf{D})$	
$(\mathbf{A}\otimes\mathbf{B}):(\mathbf{C}\times\mathbf{D}) = \mathbf{A}\otimes(\mathbf{D}^*\,\mathbf{B}^*\,\mathbf{C})$ $(\mathbf{A}\otimes\mathbf{B}){\bullet}{\circ}(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\,\mathbf{C}\,\mathbf{B})\times\mathbf{D}$ $(\mathbf{A}\otimes\mathbf{B}){\circ}{\bullet}(\mathbf{C}\times\mathbf{D}) = \mathbf{C}\times(\mathbf{A}^*\,\mathbf{D}\,\mathbf{B}^*)$	$(\mathbf{A}\times\mathbf{B}):(\mathbf{C}\otimes\mathbf{D}) = (\mathbf{A}\,\mathbf{C}^*\,\mathbf{B}^*)\otimes\mathbf{D}$ $(\mathbf{A}\times\mathbf{B}){\bullet}{\circ}(\mathbf{C}\otimes\mathbf{D}) = \mathbf{A}\times(\mathbf{C}^*\,\mathbf{B}\,\mathbf{D}^*)$ $(\mathbf{A}\times\mathbf{B}){\circ}{\bullet}(\mathbf{C}\otimes\mathbf{D}) = (\mathbf{C}\,\mathbf{A}\mathbf{D})\times\mathbf{B}$
$(\mathbf{A}\times\mathbf{B}):(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\,\mathbf{D})\boxtimes(\mathbf{B}\,\mathbf{C})$ $(\mathbf{A}\times\mathbf{B}){\bullet}{\circ}(\mathbf{C}\times\mathbf{D}) = (\mathbf{B}:\mathbf{C})\,\mathbf{A}\times\mathbf{D}$ $(\mathbf{A}\times\mathbf{B}){\circ}{\bullet}(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}:\mathbf{D})\,\mathbf{C}\times\mathbf{B}$	
$(\mathbf{A}\otimes\mathbf{B}):(\mathbf{C}\boxtimes\mathbf{D}) = \mathbf{A}\otimes(\mathbf{C}^*\,\mathbf{B}\,\mathbf{D})$ $(\mathbf{A}\otimes\mathbf{B}){\bullet}{\circ}(\mathbf{C}\boxtimes\mathbf{D}) = (\mathbf{A}\,\mathbf{C})\boxtimes(\mathbf{D}\,\mathbf{B})$ $(\mathbf{A}\otimes\mathbf{B}){\circ}{\bullet}(\mathbf{C}\boxtimes\mathbf{D}) = (\mathbf{C}\,\mathbf{B}^*)\boxtimes(\mathbf{A}^*\,\mathbf{D})$	$(\mathbf{A}\boxtimes\mathbf{B}):(\mathbf{C}\otimes\mathbf{D}) = (\mathbf{A}\,\mathbf{C}\,\mathbf{B}^*)\otimes\mathbf{D}$ $(\mathbf{A}\boxtimes\mathbf{B}){\bullet}{\circ}(\mathbf{C}\otimes\mathbf{D}) = (\mathbf{A}\,\mathbf{D}^*)\boxtimes(\mathbf{C}^*\,\mathbf{B})$ $(\mathbf{A}\boxtimes\mathbf{B}){\circ}{\bullet}(\mathbf{C}\otimes\mathbf{D}) = (\mathbf{C}\,\mathbf{A})\boxtimes(\mathbf{B}\,\mathbf{D})$
$(\mathbf{A}\boxtimes\mathbf{B}):(\mathbf{C}\boxtimes\mathbf{D}) = (\mathbf{A}\,\mathbf{C})\boxtimes(\mathbf{B}\,\mathbf{D})$ $(\mathbf{A}\boxtimes\mathbf{B}){\bullet}{\circ}(\mathbf{C}\boxtimes\mathbf{D}) = (\mathbf{A}\,\mathbf{D}^*)\otimes(\mathbf{C}^*\,\mathbf{B})$ $(\mathbf{A}\boxtimes\mathbf{B}){\circ}{\bullet}(\mathbf{C}\boxtimes\mathbf{D}) = (\mathbf{C}\,\mathbf{B}^*)\otimes(\mathbf{A}^*\,\mathbf{D})$	
$(\mathbf{A}\times\mathbf{B}):(\mathbf{C}\boxtimes\mathbf{D}) = (\mathbf{A}\,\mathbf{D})\times(\mathbf{B}\,\mathbf{C})$ $(\mathbf{A}\times\mathbf{B}){\bullet}{\circ}(\mathbf{C}\boxtimes\mathbf{D}) = \mathbf{A}\times(\mathbf{D}\,\mathbf{B}^*\,\mathbf{C})$ $(\mathbf{A}\times\mathbf{B}){\circ}{\bullet}(\mathbf{C}\boxtimes\mathbf{D}) = (\mathbf{C}\,\mathbf{A}^*\,\mathbf{D})\times\mathbf{B}$	$(\mathbf{A}\boxtimes\mathbf{B}):(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\,\mathbf{C})\times(\mathbf{B}\,\mathbf{D})$ $(\mathbf{A}\boxtimes\mathbf{B}){\bullet}{\circ}(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\,\mathbf{C}^*\,\mathbf{B})\times\mathbf{D}$ $(\mathbf{A}\boxtimes\mathbf{B}){\circ}{\bullet}(\mathbf{C}\times\mathbf{D}) = \mathbf{C}\times(\mathbf{B}\,\mathbf{D}^*\,\mathbf{A})$

Table 3: Double contractions of tensors of fourth order.

$(\mathbf{A}\otimes\mathbf{B})^T=\mathbf{A}^*\otimes\mathbf{B}^*$ $(\mathbf{A}\times\mathbf{B})^T=\mathbf{B}\times\mathbf{A}$ $(\mathbf{A}\boxtimes\mathbf{B})^T=\mathbf{B}\boxtimes\mathbf{A}$	$(\mathbf{A}\otimes\mathbf{B})^D=\mathbf{B}\otimes\mathbf{A}$ $(\mathbf{A}\times\mathbf{B})^D=\mathbf{B}^*\times\mathbf{A}^*$ $(\mathbf{A}\boxtimes\mathbf{B})^D=\mathbf{A}^*\boxtimes\mathbf{B}^*$
$(\mathbf{A}\otimes\mathbf{B})^{ti}=\mathbf{A}\boxtimes\mathbf{B}$ $(\mathbf{A}\times\mathbf{B})^{ti}=\mathbf{A}\times\mathbf{B}^*$ $(\mathbf{A}\boxtimes\mathbf{B})^{ti}=\mathbf{A}\otimes\mathbf{B}$	$(\mathbf{A}\otimes\mathbf{B})^{dl}=\mathbf{A}^*\otimes\mathbf{B}$ $(\mathbf{A}\times\mathbf{B})^{dl}=\mathbf{B}\boxtimes\mathbf{A}$ $(\mathbf{A}\boxtimes\mathbf{B})^{dl}=\mathbf{B}\times\mathbf{A}$
$(\mathbf{A}\otimes\mathbf{B})^{to}=\mathbf{B}^*\boxtimes\mathbf{A}^*$ $(\mathbf{A}\times\mathbf{B})^{to}=\mathbf{A}^*\times\mathbf{B}$ $(\mathbf{A}\boxtimes\mathbf{B})^{to}=\mathbf{B}^*\otimes\mathbf{A}^*$	$(\mathbf{A}\otimes\mathbf{B})^{dr}=\mathbf{A}\otimes\mathbf{B}^*$ $(\mathbf{A}\times\mathbf{B})^{dr}=\mathbf{A}\boxtimes\mathbf{B}$ $(\mathbf{A}\boxtimes\mathbf{B})^{dr}=\mathbf{A}\times\mathbf{B}$
$(\mathbf{A}\otimes\mathbf{B})^t=\mathbf{B}^*\otimes\mathbf{A}^*$ $(\mathbf{A}\times\mathbf{B})^t=\mathbf{A}^*\times\mathbf{B}^*$ $(\mathbf{A}\boxtimes\mathbf{B})^t=\mathbf{B}^*\boxtimes\mathbf{A}^*$	$(\mathbf{A}\otimes\mathbf{B})^d=\mathbf{A}^*\otimes\mathbf{B}^*$ $(\mathbf{A}\times\mathbf{B})^d=\mathbf{B}\times\mathbf{A}$ $(\mathbf{A}\boxtimes\mathbf{B})^d=\mathbf{B}\boxtimes\mathbf{A}$

Table 4: Transposition operations for tensors of fourth order.

$\mathbf{A}^\flat{}_{,\mathbf{A}^\flat}=\mathbf{I}^*\otimes\mathbf{I}$ $\mathbf{A}^\flat{}_{,(\mathbf{A}^\flat)^*}=\mathbf{I}^*\boxtimes\mathbf{I}$ $(\mathbf{A}^\flat)^*{}_{,\mathbf{A}^\flat}=\mathbf{I}^*\boxtimes\mathbf{I}$ $(\mathbf{A}^\flat)^*{}_{,(\mathbf{A}^\flat)^*}=\mathbf{I}^*\otimes\mathbf{I}$	$\mathbf{A}^\sharp{}_{,\mathbf{A}^\sharp}=\mathbf{I}\otimes\mathbf{I}^*$ $\mathbf{A}^\sharp{}_{,(\mathbf{A}^\sharp)^*}=\mathbf{I}\boxtimes\mathbf{I}^*$ $(\mathbf{A}^\sharp)^*{}_{,\mathbf{A}^\sharp}=\mathbf{I}\boxtimes\mathbf{I}^*$ $(\mathbf{A}^\sharp)^*{}_{,(\mathbf{A}^\sharp)^*}=\mathbf{I}\otimes\mathbf{I}^*$
$\mathbf{A}^\backslash{}_{,\mathbf{A}^\backslash}=\mathbf{I}\otimes\mathbf{I}$ $\mathbf{A}^\backslash{}_{,(\mathbf{A}^\backslash)^*}=\mathbf{I}\boxtimes\mathbf{I}$ $(\mathbf{A}^\backslash)^*{}_{,\mathbf{A}^\backslash}=\mathbf{I}^*\boxtimes\mathbf{I}^*$ $(\mathbf{A}^\backslash)^*{}_{,(\mathbf{A}^\backslash)^*}=\mathbf{I}^*\otimes\mathbf{I}^*$	$\mathbf{A}^/{}_{,\mathbf{A}^/}=\mathbf{I}^*\otimes\mathbf{I}^*$ $\mathbf{A}^/{}_{,(\mathbf{A}^/)^*}=\mathbf{I}^*\boxtimes\mathbf{I}^*$ $(\mathbf{A}^/)^*{}_{,\mathbf{A}^/}=\mathbf{I}\boxtimes\mathbf{I}$ $(\mathbf{A}^/)^*{}_{,(\mathbf{A}^/)^*}=\mathbf{I}\otimes\mathbf{I}$

Table 5: Identity tensors of fourth order.

$(\mathbb{E} : \mathbb{F})^T = \mathbb{E}^d : \mathbb{F}^d$ $(\mathbb{E} \bullet\circ \mathbb{F})^T = \mathbb{F}^T \bullet\circ \mathbb{E}^T$ $(\mathbb{E} \circ\bullet \mathbb{F})^T = \mathbb{F}^T \circ\bullet \mathbb{E}^T$	$(\mathbb{E} : \mathbb{F})^D = \mathbb{F}^D : \mathbb{E}^D$ $(\mathbb{E} \bullet\circ \mathbb{F})^D = \mathbb{F}^{ti\,T} \bullet\circ \mathbb{E}^{to\,T}$ $(\mathbb{E} \circ\bullet \mathbb{F})^D = \mathbb{F}^{to\,T} \circ\bullet \mathbb{E}^{ti\,T}$
$(\mathbb{E} \bullet\circ \mathbb{F})^{ti} = \mathbb{E} \bullet\circ \mathbb{F}^{ti}$ $(\mathbb{E} \circ\bullet \mathbb{F})^{ti} = \mathbb{E}^{ti} \circ\bullet \mathbb{F}$	$(\mathbb{E} : \mathbb{F})^{dl} = \mathbb{E}^{dl} : \mathbb{F}$
$(\mathbb{E} \bullet\circ \mathbb{F})^{to} = \mathbb{E}^{to} \bullet\circ \mathbb{F}$ $(\mathbb{E} \circ\bullet \mathbb{F})^{to} = \mathbb{E} \circ\bullet \mathbb{F}^{to}$	$(\mathbb{E} : \mathbb{F})^{dr} = \mathbb{E} : \mathbb{F}^{dr}$
$(\mathbb{E} : \mathbb{F})^t = \mathbb{F}^{Dd} : \mathbb{E}^{Dd}$ $(\mathbb{E} \bullet\circ \mathbb{F})^t = \mathbb{E}^t \bullet\circ \mathbb{F}^t$ $(\mathbb{E} \circ\bullet \mathbb{F})^t = \mathbb{E}^t \circ\bullet \mathbb{F}^t$	$(\mathbb{E} : \mathbb{F})^d = \mathbb{E}^d : \mathbb{F}^d$ $(\mathbb{E} \bullet\circ \mathbb{F})^d = \mathbb{F}^T \bullet\circ \mathbb{E}^T$ $(\mathbb{E} \circ\bullet \mathbb{F})^d = \mathbb{F}^T \circ\bullet \mathbb{E}^T$

Table 6: Transposition operations applied to contracted 4th-order tensors.

$\mathbb{E}^{T\,ti} = \mathbb{E}^{to\,T}$ $\mathbb{E}^{T\,to} = \mathbb{E}^{ti\,T}$ $\mathbb{E}^{t\,T} = \mathbb{E}^{T\,t} = \mathbb{E}^D$ $\mathbb{E}^{ti\,to} = \mathbb{E}^{to\,ti} = \mathbb{E}^t$	$\mathbb{E}^{D\,dl} = \mathbb{E}^{dr\,D}$ $\mathbb{E}^{D\,dr} = \mathbb{E}^{dl\,D}$ $\mathbb{E}^{D\,d} = \mathbb{E}^{d\,D} = \mathbb{E}^t$ $\mathbb{E}^{dl\,dr} = \mathbb{E}^{dr\,dl} = \mathbb{E}^d$	$\mathbb{E}^{ti\,D} = \mathbb{E}^{D\,to} = \mathbb{E}^{d\,ti} = \mathbb{E}^{to\,d}$ $\mathbb{E}^{D\,ti} = \mathbb{E}^{to\,D} = \mathbb{E}^{d\,to} = \mathbb{E}^{ti\,d}$ $\mathbb{E}^{t\,dl} = \mathbb{E}^{dr\,t}$ $\mathbb{E}^{t\,dr} = \mathbb{E}^{dl\,t}$ $\mathbb{E}^{t\,D} = \mathbb{E}^{D\,t}$ $\mathbb{E}^{t\,d} = \mathbb{E}^{d\,t}$
$\mathbb{E}^{T\,dl} = \mathbb{E}^{dl\,T} = \mathbb{E}^{dr}$	$\mathbb{E}^{T\,dr} = \mathbb{E}^{dr\,T} = \mathbb{E}^{dl}$	$\mathbb{E}^{T\,D} = \mathbb{E}^{D\,T}$ $\mathbb{E}^{T\,d} = \mathbb{E}^{d\,T} = \mathbb{E}$

Table 7: Connections between various transposition operations $(\mathbb{E}^T = \mathbb{E}^d)$.